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A NUMERICAL SOLUTION OF THE HEAT TRANSFER EQUATION

By

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FOREIGN TECHNOLOGY DIVISION

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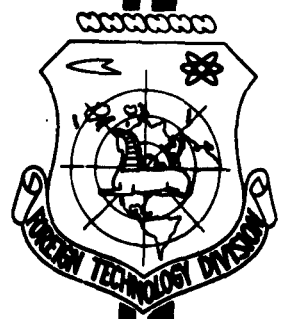
UNEDITED ROUGH DRAFT TRANSLATION

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English Pages: 64

SOURCE: Russian Book, Trudy Tbilisskogo
matematicheskogo instituta imena
A. M. Razmadze. Akademiya Nauk
Gruzinskoy SSR, Vol. 27, Izdatel'stvo
Akademii Nauk Gruz. SSR, Tbilisi, 1960,
pp. 367-410.

S/44-62-0-5-40/72

A NUMERICAL SOLUTION OF THE HEAT TRANSFER EQUATION

Sh. Ye. Mikeladze

1. Preliminary Remarks

The Cauchy problem of the propagation of heat in a one-dimensional conductor has been studied in an article by Kurant, Fridrikhs, and Levi [1]. The authors show that for a difference equation approximating a heat transfer equation to converge, the time step should decrease in proportion to the square of the spatial step. It was shown that the method may be used in solving multidimensional problems, i.e., those with two or more spatial coordinates. They do not study the law of error propagation nor do they estimate the error.

Even earlier, Richardson's work [2] had appeared in which the question of approximating the one-dimensional heat transfer problem with boundary and starting conditions was developed in one particular example with the aid of a difference equation. In this work the temperatures in the initial layer are calculated by a Fourier series and then they are found by means of a recursion relation layer by layer. The convergence of the computational process used in the work is not demonstrated nor is the error calculated.

Subsequently it proved that Richardson's computational process

diverges [2]. This fact is explained in a quite elementary manner on page 22 of an interesting survey article by P. P. Yushkov [3].

Just because of the incomplete state of the theory of approximating parabolic-type linear equations by difference equations and because of the multitude of interesting problems awaiting solution we considered it expedient to devote a few articles to this theory [4-6].

The aim of the present work is on the one hand to set forth the problem of approximating parabolic-type equations by more complete difference equations with positive coefficients guaranteeing that the computations will converge for one-dimensional and multidimensional problems under sufficiently general conditions, and on the other to point out certain findings [7-11] made more recently in a less general formulation and resulting from findings made by us earlier [5].

The main attention of this paper is devoted to deriving the general recursion relations which allow us to pass from layer to layer and also to evaluate correspondingly the error of the solution.

We worked out a method of constructing a system of linear algebraic equations relating to each other the values of the unknown functions in the nodes of any two successive layers. In the one-dimensional case it leads to simple new formulas of great accuracy; the method is also applicable to two- and three-dimensional cases which up till now have been excluded from examination. This applies especially to the three-dimensional case. It is demonstrated that the derived system of equations can be solved by iteration, the convergence of the iterative process being ensured for any intial values by appropriate selection of time and space steps from the interval or intervals of change in them. In proving the convergence

of the computational process, it is assumed that the desired solution exists.

Formulas are derived for rhombic networks. From these, in particular, are obtained the formulas for rectangular and hexagonal networks and the solution errors are estimated.

Highly accurate formulas are derived for boundary conditions of the general type.

The finite-difference equations and estimates to be derived in this work may be used equally in the Cauchy problem and in problems involving boundary and initial conditions.

In the following, for brevity, we concentrate only on the problems of heat propagation with boundary and initial conditions.

Throughout this article we shall assume that we are considering bounded, solid, uniform, isotropic bodies and will not especially stipulate this.

The length of the article does not allow us to analyze all possible cases to which our arguments are applicable. Those who have familiarized themselves with our findings will easily see how they may be used in various cases not examined in this article.

2. Error Analysis

In this section we shall investigate the errors in the various algebraic analogs of differential equations of parabolic type and deduce sufficient tests for convergence of the solution of the boundary problem for the difference analog to the corresponding solution of the parabolic differential equation.

We shall assume the values of the desired function to be known in the nodes of the first few layers and to be s in number and shall examine the difference equations which permit us to determine its

values in the succeeding layers, layer by layer.

In so doing, we shall limit ourselves to the case of a spatial variable x and time t, inasmuch as the theory which we are expounding is applicable to any number of spatial variables; but for definiteness our arguments will concern one variable. The extension to the general case presents no difficulty at all.

We shall begin with certain definitions and nomenclature permitting us to condense further arguments, and above all we shall agree, without stipulating it each time, that when $t > 0$ there exists a desired solution of $u(x,t)$ satisfying the given differential equation with boundary and initial conditions, and that all derivatives of it with respect to t and x, up to those orders which will be used below, also exist and are continuous in the closed region Q.

As for the solutions of multidimensional problems, we shall also always assume that in those regions where these solutions are considered the assumptions of the preceding paragraph hold true.

Let us take rectangle Q in the plane x,t:

$$0 \leqq x \leqq L, \quad 0 \leqq t \leqq T,$$

where T is the time interval during which the process is being studied.

Let us now examine the rectangular network

$$x = ih \left(i = 0, 1, \ldots, \frac{L}{h} \right), \quad t = kl \left(k = 0, 1, \ldots, \frac{T}{l} \right),$$

parallel to the axes of the coordinates, the sides of the cells of the network being h and l (along axes x and t).

Let us examine in the network the relationship

$$u_{\nu,s+j} = \frac{1}{\beta_{\nu,s+j}} \left(\sum_{\nu} \alpha_{\nu,s+j-1} \, u_{\nu,s+j-1} + \right.$$
$$\left. + \sum_{\mu} \alpha_{\mu,s+j-2} \, u_{\mu,s+j-2} + \dots + \sum_{\tau} \alpha_{\tau,j} \, u_{\tau,j} \right) +$$
$$+ R_{\nu,s+j} \left(j = 0, 1, \dots, \frac{T}{l} - s; \ s \geq 1 \right), \tag{1}$$

relating to each other the values of u(x,t) at the points (network nodes) lying on the segments

$$t = jl, \quad t = (j+1)l, \dots, t = (j+s-1)l \tag{2}$$

with its values at the points (ih, (j+s) l) of the segment t = (j+s) l.

In the above relationship (1) for all the values of v, s, and j we shall have

$$v_{\nu,s+j} = u(\nu h, \ (s+j)l);$$

but the coefficients α and β are functions of the points defined in rectangle Q, while $R_{1, \ s+j}$ is the remainder.

Rejecting $R_{1,s+j}$ we arrive at the recursion equation

$$U_{i,s+j} = \frac{1}{\beta_{i,s+j}} \left(\sum_{\nu} \alpha_{\nu,s+j-1} \, U_{\nu,s+j-1} + \right. \tag{3}$$
$$\left. + \sum_{\mu} \alpha_{\mu,s+j-2} \, U_{\mu,s+j-2} + \dots + \sum_{\tau} \alpha_{\tau,j} \, U_{\tau,j} \right).$$

permitting us to find very easily the successively approximated values of the desired function in all the nodes lying within Q, starting from the values on the initial segments derived from Eqs. (2) when j = 0.

Let us now investigate in what cases, when $l \to 0$, the approximate values of $U_{1, \ s+j}$ will tend to the exact values $u_{1, \ s+j}$ in the nodes of the network.

Using Formulas (1) and (3), we can write

$$\xi_{i,s+j} = A_{i,s+j} \frac{\sum_\nu \alpha_{\nu,s+j-1} \xi_{\nu,s+j-1} + \dots + \sum_\tau \alpha_{\tau,j} \xi_{\tau,j}}{\beta^*_{i,s+j}} + \qquad (4)$$
$$+ R_{i,s+j} \left(j = 0, 1, \dots, \frac{T}{l} - s \right);$$

where

$$\xi_{i,s+j} = u_{i,s+j} - U_{i,s+j}$$

is the error, and

$$A_{i,s+j} = \frac{\beta^*_{i,s+j}}{\beta_{i,s+j}} \quad (\beta_{i,s+j} \neq 0), \qquad (5)$$
$$\beta^*_{i,s+j} = \sum_\nu \alpha_{\nu,s+j-1} + \sum_\mu \alpha_{\mu,s+j-1} + \dots + \sum_\tau \alpha_{\tau,j}.$$

Let M and δ respectively designate the greatest values of
$|R_{1,s+j}|$ and $|A_{1,\ s+j}|$ in the rectangle Q; let us now limit ourselves
to the assumptions that coefficients α are not negative, the sums
of $\beta^*_{1,\ s+j}$ having positive values for all

$$\alpha_{\nu,s+j-1}, \quad \alpha_{\mu,s+j-1}, \dots, \quad \alpha_{\tau,j}$$

not equal simultaneously to 0.

We shall examine the absolute values of the errors in the nodes
of the initial segments $\underline{t}$ = 0, l,..., (s-1)l and the sides of rectangle
Q: x = 0, x = L; we shall designate the largest one of them by ε.
We shall designate by ϑ_{s+j} the upper bound of the absolute values
of the errors which occur on the layer (s+j)l as result of rounding
off the values of $U_{1,\ s+j}$ computed with the aid of Formula (3). We
shall show that Formula (4) will allow us to investigate the complete
error $\xi_{1,\ s+j}$ determined by the errors in the initial values, the
error in the Formula $R_{1,\ s+j}$, and the rounding-off errors.

Indeed, the fraction

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$$\frac{\sum_{\nu} \alpha_{\nu,\,s+j-1}\,\xi_{\nu,\,s+j-1} + \sum_{\mu} \alpha_{\mu,\,s+j-2}\,\xi_{\mu,\,s+j-2} + \ldots + \sum_{\tau} \alpha_{\tau,\,j}\,\xi_{\tau,\,j}}{\sum_{\nu} \alpha_{\nu,\,s+j-1} \qquad + \sum_{\mu} \alpha_{\mu,\,s+j-2} \qquad + \sum_{\tau} \alpha_{\tau,\,j}}$$

is the average for the errors

$$\xi_{\nu,\,s+j-1},\ldots,\xi_{\tau,\,j},$$

since all the α's are positive and do not vanish simultaneously. Hence, when $j = 0$, the absolute value of the fraction in which we are interested is not greater than ε . Therefore Formula (4) on layer $t = sl$ leads to the following estimate of the error

$$|\xi_{\nu,\,s}| \leq \varepsilon \delta + M + \vartheta_s.$$

Setting $\underline{j} = 1$ in (4) and repeating the above reasoning, we shall verify that for errors in the nodes of the segment $t = (s+1)\,l$ there exists the estimate:

$$|\xi_{\nu,\,s+1}| \leq \varepsilon \delta^2 + (1 + \delta)\,M + \vartheta_s \delta + \vartheta_{s+1}$$

and, in general, for the values of the errors on layer $t = (s+1)\,l$ we obtain:

$$|\xi_{\nu,\,s+j}| \leq \varepsilon \delta^{j+1} + (1 + \delta + \delta^2 + \ldots + \delta^j)\,M + \vartheta_{s+j} + \vartheta_{s+j-1}\delta + \ldots + \vartheta_s \delta^j.$$

We shall examine these cases separately

$$\delta < 1, \quad \delta = 1, \quad \delta > 1.$$

When $\delta < 1$ we have

$$|\xi_{\nu,\,s+j}| \leq \varepsilon \delta^{j+1} + \frac{M}{1 - \delta} + \frac{\vartheta}{1 - \delta}\,. \qquad (6)$$

where ϑ is the greatest absolute value of the errors occurring because of rounding-off the values of the solution in each node of rectangle Q.

The error $\varepsilon\delta^{J+1}$, decreasing as j increases, has almost no effect on $\xi_{1,s+J}$, for sufficiently large values of j; it vanishes when $l \to 0$. The error $M(1 - \delta)^{1}$ depends on the remainder term of the calculated formula (3) and δ. The value of $1 - \delta$ may, in particular, also be infinitesimally small in comparison with l. Therefore if M is an infinitesimal of higher order than $1 - \delta$, then $M(1 - \delta)^{-1}$ will also cease to affect the error $\xi_{1, s+J}$, beginning with a certain value of l. It remains to examine the effect of $\vartheta(1 - \delta)^{-1}$ on $\xi_{1, s+J}$, i.e., examine the sensitivity of the desired solution to rounding-off errors. We may, by holding h and l fixed, make $\vartheta(1 - \delta)^{-1}$ arbitrarily small, since we are in no way restricted in our choice of ϑ. After this, it is easy to indicate to how many decimal places one must calculate using (3), so that the rounding-off error is almost imperceptible.

When $\delta = 1$ we find

$$|\xi_{i, s+j}| \leqq \varepsilon + (j + 1)(M + \vartheta).$$

Hence it follows that in the whole region Q in the case under consideration there exists the error estimate

$$|\xi_{i, s+j}| \leqq \varepsilon + \frac{T}{l}(M + \vartheta). \tag{7}$$

Thus proceeding from the values on the initial straight lines .

$$t = jl \, (j = 0, 1, \ldots, s - 1)$$

and $x = 0$, $x = L$, we shall be able to calculate with the aid of Formula (3) all the $U_{1, s+J}$'s in succession, layer by layer, and in so doing, if the initial values are approximated with an error of order

τ with respect to l and the error in Formula (3) has an accuracy of $l^{\tau+1}$, then we can always make the sum $\varepsilon + TMl^{-1}$ less than the previously prescribed magnitude if l and $\underline{b}$ decrease indefinitely and simultaneously. It remains to take into account the value of $T\psi l^{-1}$ generated by the rounding-off errors. The effect of this error, as in the preceding case, may be made imperceptible, if it is calculated with superfluous decimal places.

Finally, for $\delta > 1$ we obtain the inequality

$$|\xi_{1,\,s+j}| \leqq \left(\bullet + \frac{M + \Phi}{\delta - 1} \right) \delta^{\frac{T}{\delta}} - \frac{M + \Phi}{\delta - 1}. \tag{8}$$

from which it is evident that if l tends to zero, then, in order for the computational process to converge, the magnitude of $\frac{T}{\delta l}$ must be limited and, in addition, ε and $(M + \Phi)(\delta - 1)^{-1}$ must be arbitrarily small [in the best case they decrease equally rapidly, i.e., they have the same orders of smallness with respect to l (or $\underline{h}$), when l and $\underline{h}$ decrease simultaneously within limits].

Thus the presence of errors in the initial values and the effect of rounding-off errors affect the final result most in the third case. In this case no matter how small the initial errors are, the right side of (8) may at times become arbitrarily large.

Therefore we shall limit ourselves in all that follows basically to the cases where $\delta \leqslant 1$). The question of decreasing the rounding-off error has been examined above and, in general, will not be examined further. In other words, we shall assume that the rounding-off errors may be made insignificantly small, for all practical purposes, by taking enough decimal places. Consequently, we shall henceforth write out the estimates of $\xi_{1,\ s+j}$, and shall at times omit the rounding-off errors $\Phi(1 - \delta)^{-1}$ and $T\psi l^{-1}$.

The limitation which the choice of the spatial interval imposes on the size of the time interval for obtaining a convergent computational process will be stated later.

3. General Linear Nonhomogeneous Equation of the Second Order with Two Variables

Let us examine the differential equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu + g \quad (a > 0). \tag{9}$$

and seek the function $u(x,t)$, which within the rectangle Q (of the preceding section) will satisfy Eq. (9) and on the border of Q, will satisfy the initial condition

$$u(x,0) = \varphi(x) \tag{10}$$

and the boundary conditions

$$u(0,t) = f_1(t), \quad u(L,t) = f_2(t). \tag{11}$$

We shall assume, for generality, that the coefficients a,b,c, and g are continuous functions of the point (x,t) in Q; we shall assume that the functions $\varphi(x)$, $f_1(t)$, $f_2(t)$ are also continuous in the corresponding intervals

$$0 \leqq x \leqq L \text{ and } 0 \leqq t \leqq T.$$

In addition we shall assume that

$$\varphi(0) = f_1(0) \text{ and } \varphi(l) = f_2(0).$$

Equation (9) may be replaced [5] by the finite-difference equation

$$U_{i,t+1} = \alpha_{i,t} U_{i,t} + \alpha_{i+1,t} U_{i+1,t} + \alpha_{i-1,t} U_{i-1,t} + lg_{i,t} \tag{12}$$

Estimates suitable for investigating convergence can be obtained almost at the outset by determining the coefficients α of Formula (12) in conformity with (4) and then by finding M, $\beta^*_{i, \ s+j}$ and δ. For the coefficients of α we obtain the formulas

$$\alpha_{i, k} = 1 - \frac{2 l a_{i, k}}{h^2} + l c_{i, k},$$

$$\alpha_{i+1, k} = \frac{l a_{i, k}}{h^2}\left(1 + \frac{h}{2} \frac{b_{i, k}}{a_{i, k}}\right),$$

$$\alpha_{i-1, k} = \frac{l a_{i, k}}{h^2}\left(1 - \frac{h}{2} \frac{b_{i, k}}{a_{i, k}}\right).$$

Here the previous nomenclature is kept; thus, for example $\alpha_{1, \ k}$ is the value of α in the node (1h, kl).

Then we obtain for (12)

$$M = \frac{l}{12}\left(6 l M_2 + h^2 M_4 a_{max} + 2 h^2 M_3 |b|_{max}\right), \qquad (13)$$

where M_2, M_3, and M_4 designate, respectively, the maximum absolute values of the derivatives

$$\frac{\partial^2 u(x, t)}{\partial t^2}, \quad \frac{\partial^3 u(x, t)}{\partial x^3}, \quad \frac{\partial^4 u(x, t)}{\partial x^4}$$

in Q. Further, by using Formula (5) we obtain

$$\beta_{i, \ s+j} = 1, \quad \beta^*_{i, \ s+j} = 1 + l c_{i, \ s+j},$$
$$\delta = 1 + l |c|_{max},$$
$$\delta^{\frac{T}{l}} < e^{T|c|_{max}}$$

and estimate (8) in this case assumes the form

$$|\xi_{i, \ s+j}| < \varepsilon \, e^{T|c|_{max}} + (e^{T|c|_{max}} - 1) \frac{M + \delta}{l|c|_{max}}, \qquad (14)$$

where M is given by Formula (13), and ε designates the upper limit of the absolute values of u(x, t) in the nodes of the network lying on the sides of the rectangle Q:

$$t = 0, \quad x = 0, \quad x = L.$$

To ensure the validitiy of the estimate obtained (14), l and h should be chosen so that

$$l - \frac{2l\, a_{i,k}}{h^2} + l c_{i,k} \geqq 0,$$

$$1 + \frac{h}{2}\, \frac{b_{i,k}}{a_{i,k}} > 0,$$

$$1 - \frac{h}{2}\, \frac{b_{i,k}}{a_{i,k}} > 0,$$

in all the nodes lying within the rectangle Q. The last two in-equalities occur for all values of a and b and small values of h. If we are interested in a value of l for some specific value of $\underline{h}$, then for convergence of the computational process we must take

$$l \leqq \frac{h^2}{2a_{max} + h^2 |c|_{max}}$$

provided that

$$2a_{i,k} - h^2 c_{i,k} > 0$$

in all, the internal nodes of Q.

Now let the coefficients of Eq. (9), a, b, and c, be constant while $c \leqslant 0$. Then

$$\beta^*_{i,n+j} = 1 - l|c|; \quad \hat{\beta}_{i,n+j} = 1,$$

and, consequently, when $c < 0$ we are dealing with the first case of section 2; but when $c = 0$ we have the second case. Therefore if the calculations are performed with the aid of the formula

$$U_{i,k+1} = \frac{la}{h^2} \left\{ \left[-2 + \frac{h^2}{la}(1 + lc) \right] U_{i,k} + \left(1 + \frac{hb}{2a} \right) U_{i+1,k} + \right.$$

$$\left. + \left(1 - \frac{hb}{2a} \right) U_{i-1,k} \right\} + l\, g_{i,k},$$

then in order to obtain an estimate suitable for any $c \leqslant 0$ we must

use (7), although for $c < 0$, in general, Estimate (6) is more advantageous.

Accordingly we obtain

$$|\xi_{i,\,s+j}| \leq \epsilon + \frac{T}{12}\,(6lM_3 + h^2 a M_4 + 2h^2|b|M_3 + 12\phi l^{-1})$$

provided that

$$\frac{h|b|}{a} < 2, \quad l \leq \frac{h^2}{2a + h^2|c|}.$$

Thus

$$|\xi_{i,\,s+j}| \leq \epsilon + \frac{T}{12}\left[\left(\frac{6M_3}{2a + h^2|c|} + aM_4 + 2|b|M_3\right)h^2 + \frac{12\phi}{l}\right].$$

Finally let us examine the differential equation of heat propagation of a thin thermally insulated rod of length L (with coefficient of heat conductivity a^2):

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{15}$$

under Initial Condition (10) and Boundary Conditions (11).

The finite-difference equation approximating this equation has the form

$$U_{i,\,s+1} = \left(1 - 2\frac{la^2}{h^2}\right)U_{i,\,s} + (U_{i-1,\,s} + U_{i+1,\,s})\frac{la^2}{h^2}. \tag{16}$$

The computational process converges when l and $\underline{h}$ satisfy the equation

$$\frac{la^2}{h^2} \leq \frac{1}{2}. \tag{17}$$

The estimate of the error in this case has the form:

$$|\xi_{i,\,s+j}| \leq \epsilon + T\left(\frac{a^2 M_4 h^2}{3} + \frac{\phi}{l}\right).$$

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4. Numerical Solution of Heat-Conduction Equation

In this section we shall examine an improved difference method of solving the general differential equation of heat conduction

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} - bu \ (0 \leq x \leq L; \ 0 \leq t \leq T), \qquad (18)$$

where $\underline{b}$ is a positive constant and $u(x,t)$ denotes the temperature.

We arrive at Eq. (18) when studying the distribution of heat in a bounded rod $(0, L)$, if we take into consideration the transfer of heat to the external space. The case $b = 0$ (i.e., the case of the thermally insulated column) leads to Differential Equation (15). Difference equations of high accuracy approximating (15) were studied in another work [5]; later a whole series of works was devoted to them among which we may note some listed in the references [7-13].

Although Eq. (18) is brought to the form (15) by substituting $u = e^{-bt}v$ into it, it would nevertheless be incorrect to neglect the problems of direct approximation of the general equations by using difference equations of high accuracy.

The basic question which arises here consists in finding the conditions under which the computational process converges, since the method of approximating the differential equation by a difference equation remains the same as before [5].

Convergence occurs if, in addition to the limitations of Section 2, the product lb (l as before designates the spacing with respect to $\underline{t}$) is changed in a special segment ensuring a bounded change in $\lambda = h^2/l a^2$, where $\underline{h}$ is the spacing with respect to $\underline{x}$. An exact enumeration of the conditions under which the process converges is given later.

In order to derive an equation of great accuracy let us expand

the differences $u_{i+1, k} - u_{i, k}$ and $u_{i-1, k} - u_{i, k}$ in accordance
with Taylor's formula, set up the expression $u_{i+1, k} - 2u_{i, k} + u_{i-1,k}$,
and replace in it the derivatives with respect to $\underline{x}$ according to the
formulas

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\left(\frac{\partial u}{\partial t} + bu\right),$$

$$\frac{\partial^4 u}{\partial x^4} = \frac{1}{a^4}\left(\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial u}{\partial t} + b^2 u\right),$$

the lower one of which is obtained by differentiating the upper one and
by expressing the derivatives contained in the derived equation in
terms of the right side of the upper one and the derivatives of
lower order.

We find

$$u_{i+1, k} + u_{i-1, k} - \left(2 + \frac{h^2 b}{a^2} + \frac{h^4 b^2}{12 a^4}\right)u_{i, k} =$$

$$= \left(\frac{h^2}{a^2} + \frac{bh^4}{6a^4}\right)\frac{\partial u_{i, k}}{\partial t} + \frac{h^4}{12a^4}\frac{\partial^2 u_{i, k}}{\partial t^2} + \frac{h^6}{360}\frac{\partial^6 u(\xi, t)}{\partial \xi^6}$$

$$(i-1)h < \xi < (i+1)h.$$

Let us now expand the differences $u_{i, k+1} - u_{i, k}$, $u_{i, k+2} -$
$- u_{i, k}$ according to the same Taylor formula, multiply them respect-
ively by the arbitrary, as yet undetermined factors α and β, and add.
We obtain

$$\alpha u_{i, k+1} + \beta u_{i, k+2} - (\alpha + \beta) u_{i, k} = (\alpha + 2\beta) l \frac{\partial u_{i, k}}{\partial t} +$$

$$+ \left(\frac{\alpha}{2} + 2\beta\right) l^2 \frac{\partial^2 u_{i, k}}{\partial t^2} + \frac{\alpha l^3}{6}\frac{\partial^3 u(x, \eta)}{\partial \eta^3} + \frac{4\beta l^3}{3}\frac{\partial^3 u(x, \zeta)}{\partial \zeta^3}.$$

Subtracting the two latter equations term by term we obtain a
relationship permitting us to write

$$\alpha u_{i, k+1} + \beta u_{i, k+2} - (u_{i+1, k} + u_{i-1, k}) +$$

$$+ \left(2 - \alpha - \beta + \frac{h^2 b}{a^2} + \frac{h^4 b^2}{12a^4}\right)u_{i, k} = - R_{i, k};$$ (19)

where

$$\alpha = 2\frac{h^2}{la^2} - \frac{h^4}{6l^2a^4}(1 - 2lb),$$

$$\beta = (1 - lb)\frac{h^4}{12\,l^2a^4} - \frac{h^2}{2la^2},$$ (20)

the remainder term assuming the form

$$R_{i,k} = \frac{\alpha l^2}{6}\frac{\partial^3 u(x,\eta)}{\partial \eta^3} + \frac{4\beta l^2}{3}\frac{\partial^3 u(x,\zeta)}{\partial x^3} - \frac{h^6}{360}\frac{\partial^6 u(\xi,t)}{\partial \xi^6}.$$ (21)

In addition, we find

$$\beta^*_{i,j} = \beta - \frac{h^2 b}{a^2} - \frac{h^4 b^2}{12a^4}.$$

In order for the computational process to converge, we require that the quantities α, β, $\beta^*_{i,j}$ and the coefficients of $u_{i,k}$ satisfy the inequalities

$$\alpha \leq 0, \quad \beta > 0, \quad \beta^*_{i,j} > 0,$$

$$\alpha + \beta \geqq 2 + \frac{h^2 b}{a^2} + \frac{h^4 b^2}{12a^4}.$$

In this case, $\beta^*_{i,j} \leqslant \beta$, i.e., $\delta \leqslant 1$ and consequently Estimates (6) and (7) will occur, respectively.

We thus obtain inequalities of the form

$$6 - (1 - \Delta)\lambda < 0,$$
$$12 - (1 - 2\Delta)\lambda \leqq 0,$$
$$6 + 12\Delta - (1 - \Delta - \Delta^2)\lambda < 0,$$
$$24 - (18 - 12\Delta)\lambda + (1 - 3\Delta + \Delta^2)\lambda^2 \leqq 0,$$ (22)

where

$$\Delta = lb, \quad \lambda = \frac{h^2}{la^2}.$$ (23)

Since we are interested in the case $b \geqslant 0$, we should attribute only non-negative values to Δ. On the other hand, λ should be bounded. Therefore Δ should assume values from the interval bounded

by O and the lowest positive root from the number of roots of the following equations:

$$1 - 2\Delta = 0, \quad 1 - \Delta = 0, \quad 1 - \Delta - \Delta^2 = 0, \quad 1 - 3\Delta + \Delta^2 = 0.$$

By calculating we find that this root is equal to 0.3819660..., and consequently l must be such that

$$0 \leqq \Delta < 0.3819660... \tag{24}$$

and it is obvious that if Δ and λ are chosen so that the second of the inequalities from the top in (22) is satisfied, the first inequality in (22) will also be satisfied.

Thus the first inequality can be omitted and the possibility of the remaining three inequalities for Δ's defined on segment (24) may be investigated. Thus we should investigate the functions defined by the inequalities

$$F_1(\Delta, \lambda) \equiv 12 - (1 - 2\Delta)\lambda \leqq 0,$$
$$F_2(\Delta, \lambda) \equiv 6 + 12\lambda - (1 - \Delta - \Delta^2)\lambda < 0,$$
$$F_3(\Delta, \lambda) \equiv 24 - (18 - 12\Delta)\lambda + (1 - 3\Delta + \Delta^2)\lambda^2 \leqq 0.$$

Investigation shows that in the region D bounded by the segments

$$\Delta = 0, \quad \Delta = 0.3819660...$$

and the curves

$$\lambda = \frac{12}{1 - 2\Delta},$$

$$\lambda = \frac{9 - 6\Delta + \sqrt{57 - 36\Delta + 12\Delta^2}}{1 - 3\Delta + \Delta^2},$$

the functions F_1, F_2, and F_3 have negative signs and that

$$\frac{12}{1 - 2\Delta} \leqq \lambda \leqq \frac{9 - 6\Delta + \sqrt{57 - 36\Delta + 12\Delta^2}}{1 - 3\Delta + \Delta^2} \tag{25}$$

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for any Δ of (24) and any point (Δ, λ) lying in region D.

When $\Delta = 0$, i.e., when $b = 0$, we hence obtain, in particular, that for the computational process to converge for (15) realized with the help of Formula (19) the values of $\lambda = h^2/la^2$ should be taken from the interval

$$12 \underset{=}{\overset{.}{\smile}} \lambda \underset{=}{\overset{.}{\smile}} 9 + \sqrt{57}.$$

This inequality was obtained by another method by P. P. Yushkov [12].

The estimate of the error in approximating (18) by the difference equation [obtained from another article [19] after neglecting the remainder term] has the form

$$|\xi_{n,n+j}| \leq \varepsilon + \frac{T}{6\mu l}\left[(|\alpha| + 8\beta) l^2 M_3 + \frac{h^4}{60}M_6\right]. \qquad (26)$$

where M_3 and M_6 designate, respectively, the greatest absolute values of the partial derivatives

$$\frac{\partial^3 u(x, t)}{\partial t^3} \quad \text{and} \quad \frac{\partial^6 u(x, t)}{\partial x^6}$$

in the rectangle Q, and ε is the greatest value of the absolute error in the initial values of $u(x, t)$ in the nodes of the segments

$$t = 0, \quad t = l, \quad x = 0, \quad x = L.$$

Let us note, moreover, that the values of $u(x,t)$ in the nodes of the segments

$$t = 0, \quad x = 0, \quad x = L$$

are known to us from the boundary conditions. The values of $u(x, t)$ in the nodes of the segment $t = l$ may be calculated, for example, by using Taylor's formula [5, 13].

Hence the following conclusion:

Theorem. If $\Delta = lb$ and $\lambda = h^2/la^2$, during a change of h and λ, satisfy Inequalities (24) and (25), then Eq. (18) with Initial

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Condition (10) and Boundary Conditions (11) may be solved numerically with the aid of Formula (19) without the remainder term $R_{i, k}$. The error $\xi(x, t)$ in each internal node of rectangle Q at the moment $t = T$ will satisfy Inequality (26).

In (19) and (20) let us now assume $\beta = 0$ and transform the formulas obtained by using relationships (23). Then the finite-difference equation approximating differential Eq. (8) is transformed into

$$U_{i, k+1} = \frac{(1 - \Delta)^2}{6} (U_{i+1, k} + U_{i-1, k}) + \frac{\Delta^2 - 2\Delta + 4}{6} U_{i, k} \quad (\Delta < 1); \qquad (27)$$

the error in approximating this formula is

$$-\frac{(1 - \Delta)^2}{6} R_{i, k} = -\frac{(1 - \Delta)^2 h^4}{1296 a^4} \frac{\partial^4 u(x, \eta)}{\partial \eta^2} + \frac{(1 - \Delta)^2 h^6}{2160} \frac{\partial^6 u(\xi, t)}{\partial \xi^6},$$

so that

$$M = \left[\frac{(1 - \Delta)^3 M_3}{1296 a^4} + \frac{(1 - \Delta)^2 M_6}{2160} \right] h^4.$$

By computation we verify that for (27)

$$\beta_{i, k+j} = 1,$$
$$\beta'_{i, k+j} = A_{i, k+j} = 0{,}5 \Delta^2 - \Delta + 1 > 0.$$

But, since $A_{1, 8+j}$ has a maximum at the point $\Delta = 0$ and equals unity, then $\delta = 1$. Thus, if we solve the general heat-conduction equation with the aid of (27), we arrive at Estimate (7), where ε is the largest absolute error in the initial data given on the segments $t = 0$, $x = 0$, and $x = L$.

The following rule for the numerical solution of Eq. (18) with Conditions (10) and (11) stems from the preceding arguments:

Rule. To solve Eq. (18) numerically, the values of $U(x, t)$ are calculated successively, layer by layer, with the aid of recursion formula

$$U_{i, k+1} = \frac{6}{\lambda^2} \left[U_{i-1, k} + \left(1 + \frac{\lambda^2}{12} \right) U_{i, k} + U_{i+1, k} \right] \quad (k = 1, 2, \ldots),$$

where

$$\lambda = \frac{6}{1-\Delta} = \frac{h^2}{la^2} \quad (0 \leq \Delta < 1),$$

and also the values of $U_{1,1}$ in the nodes of the first layer are found from the values of $U_{1,0}$, in the nodes of the zero (initial) layer, known from (10). For the error which occurs from using this recursion formula at the moment $t = T$ the following estimate holds true

$$|\xi(x,t)| \leq \varepsilon + a^2 T \left[\frac{(1-\Delta)^2 M_3}{216 \, a^6} + \frac{(1-\Delta) M_6}{360} \right] h^4.$$

The preceding rule can be formulated as follows when $\Delta = 0$:

For a numerical solution of Eq. (15) with Conditions (10) and (11) the values of $u(x, t)$ are calculated in the nodes of the network, layer by layer, with the aid of the formula

$$U(x, t+l) = \frac{U(x-h,t) + 4U(x,t) + U(x+h,t)}{6}, \quad (28)$$

and the inequality

$$|\xi(x,t)| \leq \varepsilon + T \left(\frac{a^2 M_6 h^4}{135} + \frac{\delta}{l} \right), \quad (29)$$

ensuing stemming from (7) allows us to estimate the error in the numerical solution of Eq. (15) with the aid of (28) in any node Q at any moment $t \leq T$.

There is one more conclusion from formula (28). We arrive at this formula by setting $\lambda = 6$ in (16), inequality (17) being satisfied at this value of λ, and, consequently, the computational process converges. But now we can no longer rightfully assert that at the moment $\underline{t}$ the error resulting from rejecting the remainder term is infinitesimally small and has at least the fourth order of smallness with respect to $\underline{h}$, as follows from (29). This is a disadvantage of this conclusion.

When we finish the present section, we shall try to reduce the description of the derivation of the high-accuracy difference formulas for (15) to the widest limits of generality.

Using the expansions u(x, t) according to Taylor's formula both with respect to both $\underline{x}$ and with respect to $\underline{t}$, it is possible, by repeating the author's arguments [5, p. 83], to derive the formula

$$A_1 U_{i,\,k+1} + A_2 U_{i,\,k+2} + \ldots + A_n U_{i,\,k+n} - U_{i+1,\,k} - U_{i-1,\,k} +$$
$$+ (2 - A_1 - A_2 - \ldots - A_n) U_{i,k} = - R_{i,\,k}$$

where the coefficients A_1, $A_2, \ldots$, A_n satisfy the equations

$$A_1 + 2 A_2 + \ldots + n A_n = \lambda,$$
$$A_1 + 2^2 A_2 + \ldots + n^2 A_n = 2 \frac{2!}{4!} \lambda^2,$$
$$\cdots \cdots \cdots \cdots \cdots$$
$$A_1 + 2^n A_2 + \ldots + n^n A_n = 2 \frac{n!}{(2n)!} \lambda^n.$$

For the remainder term $R_{1,\,k}$ we obtain the estimate

$$|R_{i,\,k}| \leqq \left[2 + \frac{(2n+2)!}{\lambda^{n+1}(n+1)!} \sum_{j=1}^{n} k^{n+1} |A_i| \right] \frac{M_{2n+2}\, h^{2n+2}}{(2n+2)!}$$

where, as before, $\lambda = h^2 / la^2$.

The system found above, where $\lambda \neq 0$, has a solution, since its determinant is nothing other than the product n! times the Vandermonde determinant, of order $\underline{n}$, composed of the numbers 1, 2,..., $\underline{n}$, and therefore it is different from zero.

It would be interesting to find the conditions which must be imposed on λ and at which the estimates in Section 2 remain valid. The criteria in Section 2 in the general case result in investigating a very large number of inequalities and therefore will hardly be practicable for investigating difference equations with a large number of coefficients.

In the solution of multidimensional problems new questions arise.

We shall therefore examine below equations with three and four independent variables.

In addition, in Section 6 we shall derive difference equations in which the desired values of u(x, t) in the nodes of the segments

$$l = kl \text{ and } t = (k-1)l(k=1, 2, \dots)$$

are related to each other by a system of linear equations with a number of equations equal to the number of unknowns.

Equations of this type, possessing great accuracy, may be used to determine the unknown values of $\underline{u}$ for each layer by solving a system of linear algebraic equations, a circumstance which is especially attractive, because it frees the calculator from a diverging computational process (to which the use of the non-investigated recursion formulas sometimes leads) but does not free him, of course, from investigating the error, in order to obtain the final result with the desired degree of accuracy.

5. Concluding Remarks

This section contains a short critical survey of certain investigations devoted to the problems of approximating a one-dimensional heat-conduction equation by means of difference equations in the direction of the method and theory elaborated by the author [4, 5].

Assuming b = 0 in Relationships (19) and (20), we obtain for the solution of the differential equation describing one-dimensional heat propagation the finite-difference euqation

$$\alpha U_{l, k+1} + \beta U_{l, k+2} - (U_{l+1, k} + U_{l-1, k}) + (2 - \alpha - \beta) U_{l, k} = 0, \qquad (30)$$

where

$$\alpha = 2\lambda - \frac{\lambda^2}{6},$$

$$\beta = -\frac{\lambda}{2} + \frac{\lambda^2}{12};$$

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the remainder term has the form (21). This same expression (30) was derived by us [5, p. 83] using the method of undetermined coefficients. Concluding the derivation there, I remarked that it is possible to make up a set of difference equations of form (30) for approximating the heat-conduction equation, given λ, then I wrote out one such formula corresponding to the case $\lambda = 16$ and said that with its aid a final result may be attained with an accuracy of h^4, if the initial values of $u(x, t)$ are known with the same or greater accuracy. The proof itself is elementary, therefore it is not given there. The course of the proof is the same as for the general linear nonhomogeneous differential equation of the second order with two variables or for an equation with three independent variables (see author's papers [4, or 5 (Section 12)].

Subsequently, in D. Yu. Panov's manual [7, p. 111] a Difference Equation (28) was derived without estimating the solution error. The derivation is made with the aid of arguments similar to ours [5, Section 12].

As is easily noted, Formula (28) ensues from (30) for $\lambda = 6$ and requires no special derivation.

P. P. Yushkov, studying Formula (30) for other purposes in his early work [12] also did not notice that Formula (28), in particular, was derived from (30); but he subsequently corrected this.

I am unconvinced that such remarks are generally desirable, but in the case in question they are apposite or at least pardonable and may be continued.

Further study of difference analogs of the unidimensional heat-conduction equation and the corresponding error is found, in particular, in Milne's monograph [8, 14] and in the works of his successors [9, 10, 11, 15].

My remarks would be somewhat incomplete, if I did not note that the proof of Eq. (28) and Estimate (29) given by the authors just mentioned is based, firstly, on a method of constructing difference analogs of differential equations developed by us previously [5] and, secondly, on a method developed in other articles [4, 5] for estimating the error resulting from the approximation of a differential equation by a difference equation.

The estimate obtained by Milne [8, p. 134; 14, p. 122] will completely coincide with ours (29), if in the latter ε and $\$l^{-1}$ are deleted, i.e., no account is taken, firstly, of the error in the initial values, and, secondly, of the rounding-off error, and the nomenclature is changed.

6. Solving the Heat-Conduction Equation with an Aid of a System of Equations

Up till now we have studied only the recursion formulas which allow us to calculate the values of $\underline{u}$ step by step. The present section is dedicated to the study of the heat-conduction equation by a finite-difference method enabling us to find the values of $\underline{u}$ for the next layer by solving a special linear system of algebraic equations.

The method will be developed for application to (15), although it is suitable for the general case.

It is true that the numerical solution of the heat-conduction equation by the method of this section is considerably more complicated than the solution using recursion formulas, but on the other hand, as has already been mentioned at the end of Section 4, it frees the calculator from having to analyze the convergence of the computational process.

Consequently, we are at times simply compelled to refrain from using certain recursion formulas which make the computation practically unsuitable (unstable) as a result of the rapid growth of the error in the solution even when the error in the initial values is imperceptible or as a result of rounding-off errors.

The proposed method of constructing new formulas for the solution of the heat-conduction equation has an independent interest, since, in the first place, it can be used for solving multidimensional heat-conduction problems, and, secondly, it requires the use of quadrature formulas, in contrast to methods which extensively use formulas of numerical differentiation or to the method of undetermined coefficients [5].

Among the many such formulas let us dwell on the following ($\underline{t}$ is considered as a parameter)

$$u(x-h, t) - 2u(x, t) + u(x+h, t) =$$
$$= \frac{h^2}{12}[u'_{x^2}(x-h, t) + 10 u'_{x^2}(x, t) + u'_{x^2}(x+h, t)] - \qquad (31)$$
$$- \frac{h^6}{240} u_{x^6}^{(6)}(\xi, t)(x-h \leqq \xi \leqq x+h),$$

as the most suitable for our purposes; it was used by the author [16] to solve eigenvalue problems.

Let us now substitute $t + l$ into (31) in place of $\underline{t}$, add the resulting formula term by term to Formula (31), and replace u_{x2} in the resulting expression by $a^{-2}u_t$. Let us transform the latter expression by means of a trapezoidal formula of closed type:

$$u'_t(x, t+l) + u'_t(x, t) = \qquad (32)$$
$$= \frac{2}{l}[u(x, t+l) - u(x, t)] + \frac{l^2}{6} u'''_{t^3}(x, \eta)(t < \eta < t+l)$$

and of two more of the same resulting from (32), by replacing $\underline{x}$ in it by x-h or x + h. Reducing the similar terms, we arrive at the

difference equation

$$(6 - \lambda) U(x - h, t + l) - (12 + 10\lambda) U(x, t + l) +$$
$$+ (6 - \lambda) U(x + h, t + l) + (6 + \lambda) U(x - h, t) +$$
$$- (12 - 10\lambda) U(x, t) + (6 + \lambda) U(x + h, t) = 0,$$

(33)

the remainder term R of which may be evaluated with the aid of the inequality

$$|R| \leqq \left(\frac{1}{120} + \frac{1}{6\lambda^3} \right) M_6 h^6,$$

where M_6 is the greatest value of $\left| U_{x^6}^{(6)}(x, t) \right|$ in rectangle Q.

If now

$$\lambda = \frac{h^2}{l a^2} \neq 6,$$

then Eq. (33) will be a trinomial relating the unknown values of U in three nodes of the layer t + l to its values belonging to layer $\underline{t}$. Similar equations may be written for each node of the layer t + l. The equations for the boundary nodes with abscissas x = h and x = L-h will be binomial.

A trinomial linear algebraic system of equations was studied by us in an earlier work [17, section 25]. There an analytical method of solving a system was developed, according to which the exact value of U(h, t) is found first, and then, using the equations of the given system,

$$U(2h, t), U(3h, t), \ldots$$

are found by successive substitutions.

The system derived above may also be successfully solved by using successive approximations. The convergence will be investigated below.

In the system consisting of equations of type (33) $\underline{x}$ and $\underline{t}$ intervals may be arbitrarily chosen, which advantageously distinguishes the numerical method examined from the preceding one (Section 4). When $\lambda = 6$ we again arrive at recursion Formula (28).

It remains to show the convergence of the method of successive approximations for (33). Let us rewrite it in the form

$$U(x, t + l) = \alpha [U(x - h, t + l) + U(x + h, t + l)] + A(x, t),\qquad (34)$$

where

$$\alpha = \frac{6 - \lambda}{12 + 10\lambda};\qquad (35)$$

the quantities $A(x, t)$ depend on the values of U of the preceding $\underline{t}$ layer; therefore they are known.

We shall consider that we have chosen (completely arbitrarily) the initial approximate values of U in the nodes belonging to the segment $t + l$. The result of substituting these values into the right side of (34) will, in general, be distinguished from $u(x, t + l)$; we shall therefore estimate the closeness of the approximation to $u(x, t + l)$.

We shall designate by $\xi(x, t + l)$ the difference between $u(x, t + l)$ and the result of the substitution. Let ε designate the maximum absolute value of the difference between the exact values of $\underline{u}$ and its initial values in the nodes. We shall estimate the difference $\xi(h, t + l)$. Since $u(0, t + l)$ is known to us, we add it to $A(x, t)$.

Thus, having designated the error in the first approximation by $\xi(x, t + l)$, we shall verify that at the moment $t + l$ the following inequality will be true for it

$$|\xi_1(h, t + l)| \leqq \alpha\varepsilon.$$

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That is,

$$|\xi_1(2h, t+l)| \leqq \alpha(1+\alpha)\epsilon.$$

Similarly

$$|\xi_1(3h, t+l)| \leqq \alpha(1+\alpha+\alpha^2)\epsilon.$$

Continuing these estimates further, we find that for any values of $\underline{k}$ the following inequality will be fulfilled

$$|\xi_1(kh, t+l)| \leqq \frac{\alpha}{1-\alpha}\epsilon,$$

if $\alpha < 1$, or, according to (35), $0 < \lambda < 6$.

Now let us estimate the closeness of the second approximation to $u(x, t + l)$. As in the preceding case

$$|\xi_2(kh, t+l)| \leqq \left(\frac{\alpha}{1-\alpha}\right)^2\epsilon.$$

Continuing these estimates futher, we find that for any values of $\underline{n}$ the following inequality is fulfilled

$$|\xi_n(kh, t+l)| \leqq \left(\frac{\alpha}{1-\alpha}\right)^n\epsilon,$$

and the error $\xi(x, t + l)$ will tend to zero

$$\lim_{n \to \infty} \xi_n(kh, t+l) = 0,$$

if $\alpha(1-\alpha)^{-1}$, i.e., if $\alpha < 0.5$. Since λ belongs to the interval (0.6) and the maximum value of α when λ varies in that interval is 0.5, then by choosing h and l so that $0 < \lambda < 6$, we ensure the convergence of the successive approximations for (34).

In order to illustrate the use of Eq. (33), let us solve numerically the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{36}$$

with the boundary conditions

That is,

$$|\xi_1(2h, t+l)| \leqq \alpha(1+\alpha)\epsilon.$$

Similarly

$$|\xi_1(3h, t+l)| \leqq \alpha(1+\alpha+\alpha^2)\epsilon.$$

Continuing these estimates further, we find that for any values of $\underline{k}$ the following inequality will be fulfilled

$$|\xi_1(kh, t+l)| \leqq \frac{\alpha}{1-\alpha}\epsilon,$$

if $\alpha < 1$, or, according to (35), $0 < \lambda < 6$.

Now let us estimate the closeness of the second approximation to $u(x, t + l)$. As in the preceding case

$$|\xi_2(kh, t+l)| \leqq \left(\frac{\alpha}{1-\alpha}\right)^2\epsilon.$$

Continuing these estimates futher, we find that for any values of $\underline{n}$ the following inequality is fulfilled

$$|\xi_n(kh, t+l)| \leqq \left(\frac{\alpha}{1-\alpha}\right)^n\epsilon,$$

and the error $\xi(x, t + l)$ will tend to zero

$$\lim_{n \to \infty} \xi_n(kh, t+l) = 0,$$

if $\alpha(1-\alpha)^{-1}$, i.e., if $\alpha < 0.5$. Since λ belongs to the interval (0.6) and the maximum value of α when λ varies in that interval is 0.5, then by choosing h and l so that $0 < \lambda < 6$, we ensure the convergence of the successive approximations for (34).

In order to illustrate the use of Eq. (33), let us solve numerically the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{36}$$

with the boundary conditions

$$u = e^{-\frac{\pi^2}{5.76}t} \quad \text{when} \quad x = 0, \quad u = 0 \quad \text{when} \quad x = 1,2$$

and the initial conditions

$$u = \cos \frac{\pi}{2,4} x$$

when t = 0.

Let us take h = 0.24, l = 0.048. Then

$$\frac{h^2}{l} = 1,2 \cdot$$

and the difference equation in which we are interested will assume the form

$$U(x, t+l) = 0,2\,[U(x-h, t+l) + U(x+h, t+l)] +$$
$$+ 0,3\,[U(x-h, t) + U(x+h, t)]. \tag{37}$$

If we successively set in it x = h, 2h, 3h, 4h and t = 0, we obtain a system of equations

$$U_{1,1} = 0,2\,U_{2,1} + 0,72692,$$
$$U_{2,1} = 0,2\,(U_{1,1} + U_{3,1}) + 0,46165,$$
$$U_{3,1} = 0,2\,(U_{2,1} + U_{4,1}) + 0,33541,$$
$$U_{4,1} = 0,2\,U_{3,1} + 0,17634,$$

the solution to which is given in Table 1. At the end of the table the values of the exact solution are given for comparison.

The values of u in the nodes of the succeeding layer t = 0.096 etc. may be calculated in similar fashion.

TABLE 1

x	Approximate values of u nodes of layer t = 0.048	Values of exact solution
0	0,92105	0,92105
0,24	0,87594	0,87597
0,48	0,74511	0,74514
0,72	0,54135	0,54138
0,96	0,28461	0,28462
1,20	0	0

In the case of more complex boundary conditions the methods described above may be used almost without change. It is necessary only to properly construct the equations, approximating the boundary conditions.

Thus in the case of two boundary conditions of the general type

$$u(0, t) \dotplus a_1 u'_x(0, t) \dotminus b_1(t),$$
$$u(L, t) \dotminus a_2 u'_x(L, t) \dotminus b_2(t). \tag{38}$$

it is necessary only to replace the partial derivatives appearing in them according to the unilateral formulas of numerical differentiation (cf. author's earlier works [17, 18]).

But, unfortunately, when integrating with the aid of recursion formulas, such a replacement often leads to considerable distortion of the solution, since in the actual calculation of the partial derivatives the already calculated values of $u(x, t)$, which were subject to error, are used.

Therefore it is desirable to find some new means for approximating the boundary conditions when solving equations with the aid of (34).

It is natural to expect that they may be obtained by using for the approximation both the given initial conditions and the given differential equations simultaneously. Let us now turn our attention to the construction of these formulas.

We shall begin with a consideration of the quadrature formula

$$u(x \dotplus h, t) - u(x - h, t) = 2hu'_x(x - h, t) + $$
$$+ \frac{2}{3} h^3 u_{x^3}(x - h, t) + \frac{4}{3} h^3 u_{x^3}(x, t) + R, \tag{39}$$

where

$$|R| \leqq \frac{2h^5}{45} M_5,$$

Here M_5 denotes the maximum value of $|u_p^5(x, t)|$ in the region Q. A derivation of this formula appears on page 1210 in one of the author's previous articles [19].

The method of constructing Eq (33) with the aid of (31) can be applied almost without change to Eq. (39). After employing trapezoidal formula (32), it is only necessary to replace $\underline{x}$ by $\underline{h}$, in order to obtain the final formula. We thus obtain the relationship:

$$\left(1+\frac{4}{3}\lambda\right)u(0, t+l) = -2h[u'_x(0, t+l) + u'_x(0, t)] + u(2h, t+l) +$$
$$+ u(2h, t) - \frac{8\lambda}{3}[u(h, t+l) - u(h, t)] - \left(1-\frac{4\lambda}{3}\right)u(0, t) + R^*,$$

(40)

the remainder term R* of which may be evaluated with the aid of the inequality

$$|R^*| \leq \frac{4h^5 M_5}{45} + \frac{h^2\lambda M_3}{3},$$

where M_3 is the maximum value of $|u_p'''(x, t)|$ in the rectangle Q, and $\lambda = \frac{h^2}{la^2}$.

Determining the partial derivatives $u'_x(0, t+l)$ and $u'_x(0, t)$ for $a_1 \neq 0$ from the upper equation of (38) and substituting them into (40) leads to the equation in which we are interested.

It is also possible to derive exactly an equation similar to (40) containing $u'_x(L, t+l)$ and $u'_x(L, t)$. and then eliminate from it the derivatives with respect to the spatial variable, employing for this purpose the lower equation in (38). It should be noted only that before we proceed to construct the formula in which we are interested we must replace $\underline{h}$ by -h in Quadrature Formula (39) and then set x = L - h in the formula thus obtained.

As an example we shall again solve Differential Equation (36) with the initial condition

$$u(x, 0) = 20 \cos \frac{\pi}{2} x \, (0 \leq x \leq 1),$$

and the boundary conditions

$$u'_x(0, t) = 0, \quad u(1, t) = 0,$$

i.e., we shall solve the problem of the cooling of a rod with a thermally insulated lateral surface, when the initial temperature distribution is known, assuming that one of its ends is thermally insulated, while the other is kept at a constant temperature 0.

Let us take a = 1, h = 0.1, and l = 1 = 1/120; we then obtain: λ = 1.2 and the difference equation will again be of type (37). By setting in it x = h, 2h, ..., 9h, successively, and t = 0, we obtain a system of nine equations in ten unknowns $u_{k,1}$ (k = 0, 1,..., 9):

$$U_{1,1} = 0,2 \, (U_{0,1} + U_{2,1}) + 11,70634,$$
$$U_{2,1} = 0,2 \, (U_{1,1} + U_{3,1}) + 11,27217,$$
$$U_{3,1} = 0,2 \, (U_{2,1} + U_{4,1}) + 10,56044,$$
$$U_{4,1} = 0,2 \, (U_{3,1} + U_{5,1}) + 9,58868,$$

the last of which satisfies the boundary condition $u\left(1, \frac{1}{120}\right) = 0$.

We obtain the missing equation after satisfying the boundary condition in (40) $u'_x(0, t) = 0$ at the points (0, 0) and $\left(0, \frac{1}{120}\right)$. Rejecting the remainder term and taking h = 0.1, we obtain the following formula

$$U_{5,1} = 0,2 \, (U_{4,1} + U_{6,1}) + 8,38082,$$
$$U_{6,1} = 0,2 \, (U_{5,1} + U_{7,1}) + 6,96659,$$
$$U_{7,1} = 0,2 \, (U_{6,1} + U_{8,1}) + 5,38082,$$
$$U_{8,1} = 0,2 \, (U_{7,1} + U_{9,1}) + 3,66255,$$
$$U_{9,1} = 0,2 \, U_{8,1} + 1,85410,$$

which is accurate up to h^5.

We shall give the results of the computations (Table 2). The values of the exact solution are given at the end of the table for comparison.

TABLE 2

Solution of Eq. (36) for nodes of the layer
t = 1/120

x	Initial values of u in the nodes of the layer t = 0	Approximate values in nodes of the layer t = 1/120	Values of the exact solution
0	20	19.59301	19.59300
0,1	19.75377	19.35174	19.35178
0,2	19.02113	18.63401	18.63405
0,3	17.82013	17.45745	17.45749
0,4	16.18034	15.85103	15.85107
0,5	14.14214	13.85432	13.85435
0,6	11.75571	11.51646	11.51648
0,7	9.07981	8.89502	8.89504
0,8	6.18034	6.05456	6.05457
0,9	3.12869	3.06501	3.06502
1,0	0	0	0

The system was solved by the iteration method.

The convergence of the iteration can be proven for the general case, i.e., for a system of equations consisting of L/h - 1 equations of type (34) and one more equation

$$U(o, t+l) = \frac{3}{3+4\lambda} \left[U(2h, t+l) - \frac{8\lambda}{3} U(h, t+l) \right] + B(h, \lambda, l),$$

which results from (40). Here B(h, λ, t) is known, since it is a function of the values of u in the preceding layer.

The following inequality will hold true for the error in the first approximation at the moment t + l

$$|\xi_1(kh, t+l)| \leq \frac{\alpha}{1-\alpha} \frac{3+8\lambda}{3+4\lambda} t \quad (k=0, 1, \ldots),$$

provided that α < 1 and consequently that 0 < λ < 6, in accordance with (35). Continuing on, we find that

$$|\xi_n(kh, t+l)| \leq \left(\frac{\alpha}{1-\alpha} \frac{3+8\lambda}{3+4\lambda} \right)^n t.$$

Thus the convergence of the interations is ensured by the condition:

$$\frac{6-\lambda}{6+11\lambda} \cdot \frac{3+8\lambda}{3+4\lambda} < 1 \left(\lambda = \frac{h^2}{l a^2} \right). \tag{41}$$

From the amount by which the left side of this inequality differs from unity it is possible to judge the rapidity with which ξ_n.

When $\lambda = 1.2$, the left side of (41) is equal to is $21/52$, and consequently there is convergence. This value of λ is not the best of the number of possible values ensuring rapid convergence of the iterations in our problem.

In order to approximate the boundary condition $u_x'(0,t) = 0$ it is also possible to use Taylor's formula, the given differential equation, and Formula (32). We obtain:

$$U_{0,1} = \frac{U_{1,0} + U_{1,1} + (\lambda - 1)\, U_{0,0}}{1 + \lambda}.$$

Then for the example considered above we arrive at the formula

$$U_{0,1} = \frac{2 U_{0,0} + 10 (U_{1,0} + U_{1,1})}{22},$$

which is accurate up to h^3.

7. Multidimensional Problems

In solving multidimensional problems questions arise which require explanations. Further examinations must be carried out separately for two-dimensional and three-dimensional space.

We shall first examine the propagation of heat in a thin plate with a contour of arbitrary shape γ. For this case we have the differential equation

$$\frac{\partial u}{\partial t} = a^2 \Delta u, \tag{42}$$

where Δ denotes the Laplacian operator:

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$\underline{u}$ is the temperature of a certain point (x, y) at the moment $\underline{t}$, and a^2 is the thermal-diffusivity coefficient.

We are thus confronted with the problem of the propagation of a temperature $u(x, y, t)$ in a finite principal region D bounded by the curve γ . The problem consists in finding a solution to Eq. (42) which satisfies the initial condition and the boundary conditions.

Let us first examine the boundary-value problem, when the initial temperature distribution in the plate is known

$$u(x, y, 0) = \varphi(x, y) \tag{43}$$

and during the whole time of observation the temperature on the edge of the plate γ is kept equal to

$$u(x, y, t)\big|_\gamma = f(x, y, t), \tag{44}$$

where $f(x, y, t)$ is a given function on the curve γ, and $\underline{x}$ and $\underline{y}$ are the coordinates of a variable point on this curve.

Equations (42) with Conditions (43) and (44) has been examined previously [4, 5]. The author [4] has examined an algebraic analog which approximates the equation with an accuracy up to h^2, where $\underline{h}$ is a side of a square of the network, and in another paper [5] an analog which approximates (42) with an accuracy up to h^4 was given. The latter analog, i.e., the improved difference equation ([5] p. 92) approximating (42), has the form:

$$
\begin{aligned}
U(x, y, t+1) = \frac{1}{36}\Big\{ &U(x+h, y+h, t) + U(x-h, y+h, t) + \\
&+ U(x-h, y-h, t) + U(x+h, y-h, t) + 4[U(x+h, y, t) + \\
&+ U(x, y+h, t) + U(x-h, y, t) + U(x, y-h, t)] + 16U(x, y, t)\Big\},
\end{aligned} \tag{45}
$$

where $\underline{x}$ and $\underline{y}$ are the coordinates of the main node of the square network, i.e., that (fixed) node (x, y) for which Equality (45) is written, and $\underline{l}$ is the spacing with respect to $\underline{t}$.

For convenience Formula (45) is presented in the form:

$$U_{0, t+l} = \frac{1}{36}\left[4\sum_{k=1}^{4} U_{k, t} + \sum_{k=5}^{8} U_{k, t} + 16 U_{0, t} \right],$$ (46)

where $\sum\limits_{k\,=\,1}^{4}$ (respectively $\sum\limits_{k\,=\,5}^{8}$) is the sum of the values of U at the moment $\underline{t}$ in the network nodes at a distance $\underline{h}$ (respectively $\sqrt{2}\,h$) from the main node O.

Considerably later, Formula (45) was derived by an operator method by W. E. Milne ([8], p. 150; [14], p. 137) and written in symbolic form with the aid of a pattern

$$U(x, y, t+l) = \frac{1}{36}\begin{array}{|c|c|c|} \hline 1 & 4 & 1 \\ \hline 4 & 16 & 4 \\ \hline 1 & 4 & 1 \\ \hline \end{array}\, U(x, y, t).$$ (47)

Then Eq. (47) was proven again ([11], p. 152; [15], p. 118) and the proof was completed with the remark that it could be conveniently used for solving the heat-conduction equation in two spatial variables on computing machines.

Hexagonal networks (these are known as triangular networks in the literature) for approximating Eq. (42) have been studied by P. P. Yushkov [20].

In none of these papers do we find an estimate of the error in the numerical solution. Therefore Eq. (46) is generalized in this section for rhombic networks, and an estimate of the error in the numerical solution of the heat-conduction equation found with the aid

of this generalized formula is given. The generalized formula leads, in particular, both to Formula (46) mentioned above and, consequently, to the identical Formula (47), and also to a countless number of other formulas which are of practical importance in certain cases.

All of these formulas are important in studying the propagation of heat in square, rhombic, and hexagonal plates, although they may also be successfully used for plates with any boundary configuration. In these cases it is only necessary to add to formulas of type (46) special formulas [6] for points adjoining the boundary. Afterwards, the method developed by the author [6] for curvilinear boundaries was used in several books [8, 14], § 65; [11], 15], § 8, 6; [21], § 12.
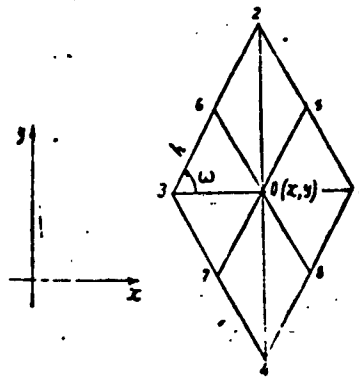
Fig. 1.

A. Let us now proceed to derive a formula for approximating (42) for the case of rhombic networks. We shall base the derivation on a previously developed method [22], § 5, which involves constructing the relationships between the values of the functions of two variables. This will permit us to write out the expansion of $u(x, y, t)$ for the main node (x, y), which lies in the center of a network rhombus with sides 2h (Fig. 1) and diagonals parallel to the coordinate axes. The expansion has the form:

$$a_0 u(x, y, t) = \sum_{i=1}^{8} a_i u_i + A_1 h^2 \Delta u(x, y, t) + A_2 h^4 \Delta^2 u(x, y, t) + R, \qquad (48)$$

where

$$A_1 = -6$$
$$A_2 = -2 \sin^2 \omega \cos^2 \omega,$$
$$a_0 = 14 + 3 tg^2 \omega + 3 ctg^2 \omega,$$

$$\sum_{i=1}^{8} a_i u_i = 4(u_5 + u_6 + u_7 + u_8) + (3 tg^2 \omega - 1) \frac{u_1 + u_3}{2} + \qquad (49)$$

$$+ (3 ctg^2 \omega - 1) \frac{u_2 + u_4}{2}, \qquad (50)$$

$$|R| \leq \frac{h^6}{45} [12 \sin^2 \omega \cos^2 \omega - 4(\sin^4 \omega + \cos^4 \omega) + (\sin \omega + \cos \omega)^6] M_6,$$

where u_1, u_2, u_3, and u_4 denote the values of $u(x, y, t)$ at the moment $\underline{t}$ at the vertices of the rhombus [with center in the main node (x, y)] with sides 2h and diagonals parallel to the coordinate axes (the odd subscripts refer to vertices lying on the horizontal diagonal; and the even, to those lying on the vertical diagonal); u_5, u_6, u_7, and u_8 denote the values of $u(x, y, t)$ at the mid-points of the sides of the same rhombus; ω is the value of the angles formed by the horizontal diagonal of the rhombus with the sides of the rhombus; and M_6 denotes (here and henceforth) the upper bound of the values of

$$\left| \frac{\partial^6 u}{\partial x^6} \right|, \quad \left| \frac{\partial^6 u}{\partial x^4 \partial y^2} \right|, \quad \left| \frac{\partial^6 u}{\partial x^2 \partial y^4} \right|, \quad \left| \frac{\partial^6 u}{\partial y^6} \right|$$

within a certain three-dimensional region of space $\underline{x}$, $\underline{y}$, and $\underline{t}$ in which the solution is sought.

In order for the coefficients of a_i to be non-negative, in all that follows ω will be determined by the condition:

$$\frac{\pi}{6} \leq \omega \leq \frac{\pi}{3}. \qquad (51)$$

From (49) it follows that

$$a_0 = \sum_{i=1}^{8} a_i. \qquad (52)$$

Then if, together with (48) and (42), we make use of the Taylor series expansions of $u(x, y, t + l)$ and $u(x, y, t + 2l)$ exactly as we did in the one-diemensional case (§4), we obtain:

$$\alpha u(x, y, t+l) + \beta u(x, y, t+2l) + (a_0 - \alpha - \beta) u(x, y, t) =$$
$$= \sum_{i=1}^{8} a_i u_i + R + \alpha R_1 + \beta R_2, \qquad (53)$$

where

$$\alpha = -2A_1\lambda + 2A_2\lambda^2 = 12\lambda - 4\lambda^2 \sin^2\omega \cos^2\omega,$$
$$\beta = 0,5\ A_1\lambda - A_2\lambda^2 = -3\lambda + 2\lambda^2 \sin^2\omega \cos^2\omega, \qquad (54)$$
$$\lambda = \frac{h^2}{l a^2},$$

while for R_1 and R_2 we have the inequalities

$$|R_1| \leq \frac{4M_6 h^6}{3\lambda^3}, \quad |R_2| \leq \frac{32M_6 h^6}{3\lambda^3}. \qquad (55)$$

Let us now choose λ such as to make β equal to zero. This will allow us to discard the term $\beta u(x, y, t + 2l)$ in (53), and we shall arrive at a recursion formula allowing us to determine the values of $\underline{u}$ at the moment $t + l$ from its values at the moment $\underline{t}$.

Moreover, we find from Eqs. (54) that

$$\alpha = \frac{9}{\sin^2\omega \cos^2\omega}, \quad \lambda = \frac{1,5}{\sin^2\omega \cos^2\omega},$$
$$a_0 - \alpha - \beta = 8 - \frac{6}{\sin^2\omega \cos^2\omega}.$$

Consequently, when $\beta = 0$

$$u(x, y, t+l) = \frac{6 - 8\sin^2\omega \cos^2\omega}{9} u(x, y, t) +$$
$$+ \frac{\sin^2\omega \cos^2\omega}{9} \sum_{i=1}^{8} a_i u_i + R_1 + \frac{\sin^2\omega \cos^2\omega}{9} R. \qquad (56)$$

It is easy to see that

$$6 - 8\sin^2\omega \cos^2\omega > 0.$$

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We also know that all the a_i's in (49) are greater than or equal to zero, if ω lies in (51). Consequently variation of ω between $\frac{\pi}{6}$ and $\frac{\pi}{3}$ entails the non-negativity of the coefficients of Eq. (56). In all that follows, therefore, let us agree to consider Formula (56) for those values of ω for which Inequality (51) is valid.

Thus in order for the computational process generated by Formula (56) to converge, it is enought that the relationship between the $\underline{h}$ and $\underline{l}$ intervals be taken such that

$$6 \leqq \lambda \leqq 8.$$

In any case, if ω belongs to (51), we obtain by using (56) and (52)

$$\beta^{\circ}_{i, s+j} = \frac{2}{3} + \frac{\sin^2\omega\cos^2\omega}{6}(a_s - 8) = 1,$$

$$\beta_{i, s+j} = 1, \quad A_{i, s+j} = 1, \quad \delta = 1,$$

and consequently the error in the numerical solution to (42) found with the aid of (56) (without the remainder term) in any node of the rhombic network at any moment $t \leqslant T$ will satisfy the inequality

$$|\xi(x, y, t)| \leqq \varepsilon + \frac{T}{l}(M + \vartheta),$$

where

$$M = |R_1| + \frac{\sin^2\omega\cos^2\omega}{9}|R|.$$

where Inequalities (50) and (55) should be fulfilled for R_1 and R.

Now let the boundary contour γ be the network contour, i.e., it consists of the nodes and links of the rhombic network. Then in the estimate just obtained ε should be tkan to mean the absolute value of the error in the function $u(x, y, t)$ $(0 \leqslant t \leqslant T)$ in the nodes of γ. Recalling the estimates for R_1 and R, we obtain the following theorem:

If the solution of Eq. (42) with Conditions (43) and (44) is sought with the aid of Formula (56) (without the remainder term), then the error in any node of the network (lying inside γ) at the moment of t + T satisfies the inequality.

$$|\xi(x, y, T)| \leq \varepsilon + \frac{Ga^2 T M_6 h^4}{270},$$

(57)

where

$$G = 160 \sin^4 \omega \cos^4 \omega + 12 \sin^2 \omega \cos^2 \omega \quad 4(\sin^6 \omega + \cos^6 \omega) +$$
$$+ (\sin \omega + \cos \omega)^6 \left(\frac{\pi}{6} \leq \omega \leq \frac{\pi}{3}\right).$$

In the case of a square network with the diagonals of the squares parallel to the coordinates axes $\omega = \frac{\pi}{4}$, i.e., $\lambda = 6$, from it we can deduce the following theorem concerning the estimate of the error in the solution:

Theorem. <u>Let the solution to Eq. (42) with Conditions (43) and (44) be sought with the aid of the formula</u>

$$U(x, y, t + l) = \frac{1}{36} [U_{1,t} + U_{2,t} + U_{3,t} + U_{4,t} +$$

(58)

$$+ 4(U_{5,t} + U_{6,t} + U_{7,t} + U_{8,t}) + 16 U_{0,t}],$$

<u>where $U_{0,t}$ is the value of U(x, y, t) at the moment t at the center of the network square.</u>

<u>Then the error in any network node (lying inside γ) at the moment t = T, in accordance with (57), satisfies the inequality</u>

$$|\xi(x, y, T)| \leq \varepsilon + \frac{2a^2 T M_6 h^4}{27},$$

(59)

<u>where h is the side of a square (inclined at an angle of 45° to the x-axis).</u>

The theorem is proven in the same way for hexagonal networks with a network contour γ consisting of the nodes and links of the

network. For such a network the following theorem applies.

Theorem. <u>If the solution to Eq. (42) with Conditions (43) and (44) is sought with the aid of the formula</u>

$$U_{0, t+1} = \frac{6U_{0, t} + U_{1, t} + U_{3, t} + U_{5, t} + U_{6, t} + U_{7, t} + U_{8, t}}{12},$$

where $U_{0, t}$ is the value of $U(x, y, t)$ at the moment <u>t</u> at the center <u>of a certain hexagon of the network, while a</u> $U_{k, t}$ <u>(k = 1, 3, 5, 6, 7, 8) is the value at its vertices, then the error in any network node (lying inside γ) at the moment t = T satisfies the inequality</u>

$$|\xi(x, y, t)| < \varepsilon + \frac{101 \, a^2 T M_6 h^4}{2160}, \tag{60}$$

where <u>h</u> is the side of a hexagon of the network.

To prove this theorem, it is enough to take $\omega = \frac{\pi}{3}$ in (56) and (57).

For the sake of completeness, let us consider a square network with the sides of the squares parallel to the coordinate axes. Let us return to Formula (48) and in it set

$$a_0 = 20, \quad A_1 = -6, \quad A_2 = -\frac{1}{2}, \quad \sum_{i=1}^{8} a_i u_i = 4 \sum_{k=1}^{4} u_k + \sum_{k=5}^{8} u_k.$$

in accordance with Formula (13) derived in chapter I (§ 2) in our earlier publication [5]. For this case Estimate (50) is already invalid. Calculation shows that it should be replaced by:

$$|R| \leq \frac{17 M_6 h^4}{45}.$$

Let us now examine the formula thus obtained, together with (53) and the left-hand equalities in (54), when

$$A_1 = -6, \quad A_2 = -\frac{1}{2}, \quad \beta = 0.$$

We then obtain $\lambda = 6$ and $\alpha = 36$, so that

$$M = \frac{M_0 h^4}{60}.$$

For these values of λ, α, and M we arrive at Formula (46), for which

$$|\xi(x, y, t)| \leqq \varepsilon + \frac{a^2 T M_0 h^4}{10}, \tag{61}$$

where $\underline{h}$ is the side of a square of the network.

As is apparent, Formula (46) and (48) have approximately the same degrees of accuracy.

Thus in view of Estimates (59), (60), and (61) we arrive at an interesting result, namely, that out of the three formulas with identical h's just examined [for the numerical solution of Eq. (42)] the formula for hexagonal networks generally yields the most accurate result.

Let us now proceed to a consideration of the general case. Let us determine α and β in (54) with the requirements that for any rhombic networks characterized by the two numbers $\underline{h}$ and ω and the time interval t, i.e., for any λ which figures in (54), the inequalities

$$-\alpha \geqq 0, \quad -a_0 + \alpha + \beta \geqq 0, \quad \beta > 0.$$

are fulfilled. In accordance with the discussions in § 2, this is necessary for the convergence of the computational process and for the estimate of the accuracy ahcieved.

Using (54), we can verify that the first and last of these three inequalities are fulfilled simultaneously when

$$\lambda \geqq \frac{3}{\sin^2 \omega \cos^2 \omega}$$

for all values of ω in the interval (51); moreover, a study of the second inequality indicates that it can be realized for any ω in (51)

when

$$\frac{9-\tau}{4\sin^2\omega\cos^2\omega} \leqq \lambda \leqq \frac{9+\tau}{4\sin^2\omega\cos^2\omega},$$

where

$$\tau = \sqrt{57-64\sin^2\omega\cos^2\omega}.$$

Furthermore, no matter what the value of ω is from (51), the last inequality may be replaced by the inequality

$$\frac{3}{\sin^2\omega\cos^2\omega} \leqq \lambda \leqq \frac{9+\tau}{4\sin^2\omega\cos^2\omega}. \tag{62}$$

Thus, λ and ω, determined by Inequalities (51) and (62), satisfy all three inequalities considered above and, consequently, guarantee the convergence of the computational process carried out for a rhombic network with the aid of Formula (53), in which it is necessary to take α and β in accordance with Formulas (54), to replace the sum $\underset{1}{\Sigma} a_1 u_1$ in accordance with (49), to substitute

$$14 + 3\,\text{tg}^2\,\omega + 3\cot^2\omega - 9\lambda + 2\lambda^2\sin^2\omega\cos^2\omega,$$

in place of the quantity $a_0 - \alpha - \beta$, and to solve the equation obtained for $u(x, y, t + 2l)$.

It remains to give estimate of the error in the numerical solution sought with the aid of the recursion formula just obtained. For this purpose, we should evaluate

$$M = \beta^{-1}(|R| + \alpha|R_1| + \beta|R_2|), \tag{63}$$

using Inequalities (50) and (55). From this there results the following theorem:

If the solution to Eq. (42) with Conditions (43) and (44) is sought with the aid of recursion Formula (53) solved relative to $u(x, y, t + 2l)$, then the error in any node of a rhombic network (lying inside γ) at the moment $t = T$ satisfies the inequality

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$$|\xi(x, y, t)| \leq \varepsilon + \frac{T}{t}(M + \vartheta),$$

provided that α and β, figuring in (63), are determined by Eqs. (54), in which λ and ω must be taken in accordance with Inequalities (51) and (62).

Here ε also denotes the maximum absolute error with which the values of $u(x, y, t)$ are calculated in the nodes of the network contour γ when $0 \leqslant t \leqslant T$ and in its internal nodes when $t = 0$ and $t = l$.

If these values are computed to an accuracy of h^4, then we obtain the values of U at the moment T with the same accuracy.

Setting $\omega = \frac{\pi}{4}$, we obtain from (62) for a square network (with the diagonals of the squares parallel to the coordinate axes) the inequality

$$12 \leqq \lambda \leqq 9 + \sqrt{41} \tag{64}$$

and the computational process carried out with the aid of the formula

$$U_{0, i+1} = (0,5\lambda^2 - 3\lambda)^{-1}[(\lambda^2 - 12\lambda)U_{0, i+1} - (0,5\lambda^2 - 9\lambda + 20)U_{0, i} +$$
$$+ \sum_{k=1}^{4} U_{k, i} + 4\sum_{k=5}^{8} U_{k, i}], \tag{65}$$

which results from (53), will converge if λ is contained in the interval of (64).

As soon as λ satisfies (64), Eq. (65) leads to values of U subject to errors determined by the inequality

$$|\xi(x, y, t)| \leqq \varepsilon + \frac{2}{9} \frac{\lambda^2 + 18\lambda - 72}{0,5\lambda^3 - 3\lambda^2} a^2 T M_6 h^4.$$

If we now set $\lambda = 12$, we may write Eq. (65) in the form

$$U_{0, i+1} = \frac{\sum_{k=1}^{4} U_{k, i} + 4\sum_{k=5}^{8} U_{k, i} + 16 U_{0, i}}{36}.$$

In this case the error $\xi(x, y, T)$ in the internal nodes of the network contour at the moment T satisfies the inequality

$$|\xi(x, y, T)| \leqq \epsilon + \frac{4a^3 T M_6 h^4}{27}.$$

The method set forth above for solving Eq. (42) is also applicable to curvilinear boundaries. In this case it is only necessary to set up and use special formulas for the points adjoining the boundary, exactly as was done in our previous publication [6, § 7].

B. Using Formulas (7) and (8) from the author's earlier paper [23], we may in a way similar to what was done for the one-dimensional case, derive a system of five-term linear algebraic equations for the numerical solution of (42). The solution of such systems requires fairly tedious work, if $\underline{h}$ is small, and the number of equations is consequently large, even if the method of successive approximations is used. However, the possibility of using the method for any spacing relationships makes it nonetheless practical. We shall therefore describe the method in its general outlines.

Let us rewrite Formula (7) of the aforementioned paper [23] in the form

$$20\,u_{0,t} = 4\sum_{k=1}^{4} u_{k,t} + \sum_{k=5}^{8} u_{k,t} - 0,5\,h^2\left[8\Delta u_{0,t} + \sum_{k=1}^{4} \Delta u_{k,t} \right], \qquad (66)$$

where $u_{k,t}$ denotes the values of u(x, y, t) in the specially selected nodes of the square network [used for (46)] at the moment $\underline{t}$. Formula (66) is accurate up to h^6.

Substituting $t + l$ into (66) in place of $\underline{t}$, we add the formula obtained to Formula (66), replace

$$\Delta u_{k,t}\,(k = 0, 1, 2, 3, 4)$$

by $a^{-2}u'_t(x_k, y_k, t)$ in the sum, and use Eq. (32), which is written for two spatial coordinates.

Thus we obtain the equation

$$(20 + 8\lambda) U_{0, t+l} - (4 - \lambda) \sum_{k=1}^{4} U_{k, t+l} - \sum_{k=5}^{8} U_{k, t+l} = \qquad (67)$$

$$= -(20 - 8\lambda) U_{0, t} + (4 + \lambda) \sum_{k=1}^{4} U_{k, t} + \sum_{k=5}^{8} U_{k, t}.$$

After writing out these equations for all the internal nodes of a square plate, we arrive at a system consisting of $\underline{n}$ linear equations in $\underline{n}$ unknown values of the function $u(x, y, t + l)$, where $\underline{n}$ is the number of internal nodes of the square bounding the plate.

The system thus obtained may be solved by the iteration method. For the proof of the convergence of the iteration process for any initial approximations of $U_{k, t+l}$ it is sufficient to require that

$$\frac{h^2}{la^2} = \lambda \leq 4.$$

In practical computations we shall assume that the values of $u_{k, t}$ already computed on the preceding layer are taken as the initial approximation for $U_{k, t+l}$.

We shall rewrite (67) in the form

$$U_{0, t+l} = a \sum_{k=1}^{4} U_{k, t+l} + b \sum_{k=5}^{8} U_{k, t+l} + A_t, \qquad (68)$$

where

$$a = \frac{4 - \lambda}{20 + 8\lambda}, \qquad b = \frac{1}{20 + 8\lambda},$$

$$A_t = b [(4 + \lambda) \sum_{k=1}^{4} U_{k, t} + \sum_{k=5}^{8} U_{k, t} - (20 - 8\lambda) U_{0, t}].$$

Thus, the quantities A_t are linear combinations of the values of U at the nodal points at the moment $\underline{t}$, combinations which shall hereafter be assumed to have been computed, i.e., the values of A_t in (68) will hereafter be assured to be known.

We shall now take the numbers $U_{k,t}$ as the initial (zero) approximation and set

$$|U_{k,t+l} - U_{k,t}| \leqq \mathbf{s.}$$

throughout the entire square region. The values of $U_{k,t+l}$ are known to us in the boundary nodes, and therefore the difference between $U_{k,t+l}$ and its zero approximations will be assumed equal to zero in the boundary nodes.

Let us now compute the values of U in the network nodes closest to the boundary of the square γ, using for this purpose the values of $\underline{u}$ given on γ at the moment $t + l$ and the initial system of values of $U_{k,t+l}$ for all the other nodes encountered in (68). In this way, at least three values of U in nodes lying on γ will figure in each of Eqs. (68). Therefore, after denoting the error of the first approximation by $\xi_1(x, y, t)$, we shall verify that at the moment $\underline{t}$ it satisfies the inequality

$$|\xi_1(x, y, t + l)| \leqq (3a + 2b)\mathbf{s} = \alpha \mathbf{s}, \tag{69}$$

where

$$\alpha = \frac{14 - 3\lambda}{20 + 8\lambda}.$$

Thus Estimate (69) holds true along the square γ_1, on which the boundary nodes lie.

Then, using the values obtained for $U_{k,t+l}$ in the nodes of γ_1, the initial system of values, and Formula (68), we calculate the values of U in the nodes of the boundary γ_2 of the square which is

next in proximity after γ_1. For the error in the first approximation at the nodes of $\dot{\gamma}_2$, the estimate

$$|\xi_1(x,y,l+l)| \leqq (a + 2b)\,\alpha\varepsilon \quad (3a + 2b)\,\varepsilon \leqq \alpha(1 + \alpha)\varepsilon.$$

will be valid. Analogously

$$|\xi_1(x,y,l+l)| \leqq (a + 2b)\,\alpha(1 + \alpha)\varepsilon + (3a + 2b)\,\varepsilon \leqq$$
$$\leqq \alpha(1 + \alpha + \alpha^2)\varepsilon,$$

where ξ_1 now denotes the error in the first approximation in the nodes of the boundary γ_3 of the square which is next in proximity.

Continuing these estimates for (68) with the non-negative coefficients $\underline{a}$ and $\underline{b}$ until all the nodes of the main square are exhausted, we verify that the estimate

$$|\xi_1(x,y,l+l)| \leqq \frac{\alpha}{1-\alpha}\varepsilon,$$

finally holds for the first approximation, since $\alpha < 1$ for any $\lambda \leqq 4$:

Using the first approximations, we calculate the second approximations, etc. by means of Formula (68) in a way similar to that used when we calculated the first approximations, etc. The error estimate for the m-th approximation has the form:

$$|\xi_m(x,y,l+l)| \leqq \left(\frac{\alpha}{1-\alpha}\right)^m \varepsilon,$$

whence follows the convergence of the successive approximations when

$$\frac{4}{7} < \lambda \leqq 4.$$

The most rapid convergence occurs when $\lambda = 4$. In this case Formula (67) assumes the simple form:

$$52\,U_{0,l+l} = \sum_{k=5}^{8} U_{k,l+l} + 8\sum_{k=1}^{4} U_{k,l} + \sum_{k=5}^{8} U_{k,l} + 12\,U_{0,l}. \qquad (70)$$

Using (70), we arrive finally at a system consisting of five-term equations relating the unknown values of $\underline{u}$ in five network nodes

at the moment $t + l$ to its nine known values at the nodes at the moment $\underline{t}$.

We obtain another formula for (42) by using Formula (8) from the author's aforementioned paper [23]. Calculation yields the formula:

$$(20 + 10\lambda)\, U_{0,t+l} = 4 \sum_{k=1}^{4} U_{k,t+l} + (1 - 0.5\lambda) \sum_{k=5}^{8} U_{k,t+l}^{\bullet} +$$
$$+ 4 \sum_{k=1}^{4} U_{k,t} + (1 + 0.5\lambda) \sum_{k=5}^{8} U_{k,t} - (20 - 10\lambda)\, U_{0,t},$$

which assumes the simple form

$$40\, U_{0,t+l} = 4 \sum_{k=1}^{4} U_{k,t+l} + 4 \sum_{k=1}^{4} U_{k,t} + 2 \sum_{k=5}^{8} U_{k,t}.$$

for $\lambda = 2$.

These formulas are accurate up to h^6. Finally it is possible to prove, although we shall not take the time to do so, however, that if the general formula of the preceding paragraph is solved for $U_0, {}_{t+l}$ and if an iteration is formed with the aid of the formula obtained, then convergence will take place for all values of λ for which

$$\frac{2}{3} < \lambda \leqq 14.$$

The method may be applied to regions with curvilinear boundaries; in these cases it is only necessary to set up and use special formulas at the points adjoining the boundary.

Thus, using Formula (6.2) in the literature [22] for the layers $\underline{t}$ and $t + l$, Eq. (42), and a formula of type (32) written for two spatial coordinates, we obtain for the boundary points the formula

$$\left(\frac{1}{h_1 h_2} + \frac{1}{h_3 h_4} + \frac{\lambda}{h^2}\right) U_{\nu, t+l} = \frac{1}{h_1 + h_2}\left(\frac{U_{1, t+l}}{h_1} + \frac{U_{2, t+l}}{h_2}\right) +$$

$$+ \frac{1}{h_3 + h_4}\left(\frac{U_{3, t+l}}{h_3} + \frac{U_{4, t+l}}{h_4}\right) + \frac{1}{h_1 + h_2}\left(\frac{U_{1, t}}{h_1} + \frac{U_{2, t}}{h_2}\right) +$$

$$+ \frac{1}{h_3 + h_4}\left(\frac{U_{3, t}}{h_3} + \frac{U_{4, t}}{h_4}\right) - \left(\frac{1}{h_1 h_2} + \frac{1}{h_3 h_4} - \frac{\lambda}{h^2}\right) U_{\nu, t},$$

.where $U_{\nu, t}$ is the value of U at the moment $\underline{t}$ at a boundary point with
the number ν, while U_1, t, U_2, t, U_3, t, U_4, t are the values of U in
the nodes separated from the main node (the node with the number ν)
by a distance of h_1 (to the left) and h_2 (to the right) in the direc-
tion of the horizontal and h_3 (upwards) and h_4 (downwards) in the
direction of the vertical, respectively; the values of the function
$u(x, y, t + l)$, denoted by $U_{\nu, t+l}$ and $U_{k, t+l}$ ($k = 1, 2, 3, 4$)
have the same meaning. As for λ, it is equal as before to $\frac{h^2}{la^2}$,
where $\underline{h}$ is the side of a square of a uniform network, so that
$h_i \ll h(i = 1, 2, 3, 4)$.

In the case of boundary conditions of the general type it is
necessary to have a formula to approximate the normal derivative.
We shall derive one such formula, which is accurate up to h^3. More
accurate formulas may be obtained by using relationships (31), (32),
and (42) for two spatial coordinates.

Let us consider the edge of a plate parallel to the y-axis,
and let us set up a relationship for the boundary condition containing
the derivative with respect to the direction of the normal perpendicu-
lar to this edge.

For this purpose, let us represent $u(x + h, y, t)$ in the form

$$u(x + h, y, t) = u(x, y, t) + h u'_x(x, y, t) + \frac{h^2}{2} u''_{x^2}(x, y, t) + R_1.$$

Expanding the functions $u(x, y - h, t)$ and $u(x, y + h, t)$ according

to Taylor's formula, we find:

$$u(x, y + h, t) - 2u(x, y, t) + u(x, y - h, t) = h^2 u''_{yy}(x, y, t) + R_g.$$

Therefore, if we discard the remainder term and use the relationships thus obtained, we may write the equality

$$2u(x + h, y, t) + u(x, y - h, t) - 4u(x, y, t) + u(x, y - h, t) =$$
$$= 2hu'_x(x, y, t) + h^2 \Delta u(x, y, t).$$

Next, let us write out the same equality for the moment $t + l$, add it to the preceding one, and then convert the relationship obtained, using for this purpose Formula (32) written for two spatial coordinates.

As a result we obtain:

$$(4 + 2\lambda) u(x, y, t + l) = -2h[u'_x(x, y, t) + u_x'(x, y, t + l)] +$$
$$+ u(x, y + h, t + l) + 2u(x + h, y, t + l) + u(x, y - h, t + l) - \tag{71}$$
$$- (4 - 2\lambda) u(x, y, t) + u(x, y - h, t) + u(x, y + h, t) + 2u(x + h, y, t).$$

If Eq. (42) is solved with the aid of Formula (70), then in (71) it is necessary to take $\lambda = 4$ and to replace the partial derivatives with respect to $\underline{x}$ in accordance with the boundary condition on the plate edge under consideration.

It is easy to derive formulas similar to (71) for the remaining three edges of the plate.

C. Let us consider the problem of the propagation of heat in a body bounded by a surface S, when the initial temperature inside the body assumes the values $f(x, y, z)$, while the boundary (the surface S bounding the body) is maintained at the temperature $\varphi(x, y, z, t)$ at all $t > 0$.

It is required to calculate the temperature of the body at each of its points at any moment of time.

The solution to the problem reduces to integration of the

the three-dimensional equation

$$\frac{\partial u}{\partial t} = a^2 \Delta u \quad \left(\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right),$$ (72)

with the boundary condition

$$u = \varphi(x, y, z, t) \text{ where } t > 0$$

and the initial condition

$$u(x, y, z, 0) = f(x, y, z).$$

Let us consider a cubic network in a space with a rectangular coordinate system x, y, z with the faces of the cubes of the network parallel to the coordinate planes.

Let us now take a cube of the network with edges 2h and its center in the node marked 0 (main node); its upper face shall be called the first square; the cross section of the cube cut by a plane passing through the point 0 parallel to the upper face, the second square; and finally the lower face shall be called the third square.

The values of u at the moment t satisfy [23] in the network nodes the relationships:

$$24 u_{0,t} = 2 \sum_{i=1}^{6} u_{i,t} + \sum_{k=7}^{18} u_{k,t} - 6 h^2 \Delta u_{0,t} - 0.5 \, h^4 \Delta^2 u_{0,t} - R,$$ (73)

$$56 u_{0,t} = 8 \sum_{i=1}^{6} u_{i,t} + \sum_{r=19}^{26} u_{r,t} - 12 \, h^2 \Delta u_{0,t} - h^4 \Delta u_{0,t} - R^*,$$ (74)

where $u_{0,t}$ is the value of u(x, y, z, t) in the main node 0; $\sum_{i=1}^{6} u_{i,t}$ is the sum of the values of u at the mid-points of the sides of the second square and in the centers of the second and third squares; $\sum_{r=19}^{26} u_{r,t}$ denotes the sum of the values of u at the vertices of the first and third squares; and, finally, $\sum_{k=7}^{18}$ is the sum of the values

of $\underline{u}$ at the mid-points of the sides of the first and third squares and at the vertices of the second square. The errors incurred from discarding the remainder terms in (73) and (74) are estimated by the inequalities:

$$|R| \leq \frac{13 \, M_6 h^8}{20}, \quad |R^*| \leq \frac{21 \, M_6 h^8}{10},$$

where M_6 is the maximum absolute value of the sixth-order partial derivatives of $u(x, y, z, t)$ inside S.

Next, we use (as was done above in the two-dimensional case), together with (73) and (74), the Taylor expansions with respect to $\underline{t}$ for

$$\alpha u (x, y, z, t-l) \text{ and } \beta u (x, y, z, t-2l),$$

where α and β are undetermined factors, and we choose α and β so as to obtain relationships which are accurate to h^6.

We thus find two general formulas. The first

$$\beta u_{0, t+2l} = - \alpha u_{0, t+l} - (24 - \alpha - \beta) u_{0, t} + 2 \sum_{i=1}^{6} u_{i, t} + \qquad (75)$$
$$+ \sum_{k=7}^{18} u_{k, t} + R_1^*,$$

where

$$\alpha = 12 \lambda - \lambda^2, \quad \beta = - 3\lambda + 0.5 \lambda^2 \left(\lambda - \frac{h^2}{la^2} \right),$$

for which we shall have

$$|R_1^*| \leq |R| + |\alpha R_1| + |\beta R_2|.$$

The second

$$\beta u_{0, t+2l} = - \alpha u_{0, t+l} - 8 \sum_{i=1}^{6} u_{i, t} - \sum_{r=19}^{26} u_{r, t} - \qquad (76)$$
$$- (56 - \alpha - \beta) u_{0, t} + R_2^*,$$

valid for any α and β such that

$$\alpha = 24 \lambda - 2 \lambda^2, \quad \beta = - 6\lambda + \lambda^2 \left(\lambda - \frac{h^2}{la^2} \right),$$

-54-

with the remainder term satisfying the inequality

$$|R_3^*| \leq |R^*| - \alpha|R_1| + |\beta R_2|.$$

In the last two inequalities the estimates

$$|R_1| \leq \frac{9 M_6 h^6}{2 \lambda^3}, \quad |R_2| \leq \frac{36 M_6 h^6}{\lambda^3}.$$

should be fulfilled for R_1 and R_2.

When $\lambda = 6$ there result as a special case from Formulas (75) and (76) the formulas examined in the literature [24]:

$$U_{0, t+1} = \frac{1}{36}\left[2 \sum_{i=1}^{6} U_{it} + \sum_{k=7}^{18} U_{k, t} + 12 U_{0, t}\right],$$

$$U_{0, t+1} = \frac{1}{72}\left[8 \sum_{i=1}^{0} U_{i, t} + \sum_{r=10}^{26} U_{..t} + 16 U_{0, t}\right].$$

The coefficients in these formulas are such that they satisfy (§ 2) all the requirements for the convergence of the computational process. It follows from this same Section 2 that the inequalities

$$|\xi| \leq \varepsilon + \frac{7 a^2 T M_6 h^4}{30},$$

$$|\xi| \leq \varepsilon + \frac{3 a^2 T M_6 h^4}{10}.$$

must hold true for the errors in these formulas at the moment T.

Such estimates can be obtained for the general formulas, if the variations in λ are limited by certain inequalities guaranteeing the non-negativity of the coefficients in the formulas under consideration.

Thus for Formula (75) the inequalities

$$-\alpha \geq 0, \quad \alpha + \beta \leq 24, \quad \beta > 0.$$

should be fulfilled.

Study of these inequalities shows that they are fulfilled simultaneously for each λ of the interval

$$12 \leq \lambda \leq 9 + \sqrt{33}.$$

In the case of Formula (76) we have

$$12 \leq \lambda \leq 14.$$

D. Here also the problem of the numerical solution of the three-dimensional heat-conduction equation may be reduced to a system of linear algebraic equations. For this, we take Eq. (73), discard the remainder term, and eliminate $\Delta^2 U_{0,t}$ from the equality obtained, using for this purpose the relationship

$$h^2 \sum_{i=1}^{6} \Delta U_{i,t} = 6h^2 \Delta U_{0,t} + h^4 \Delta^2 U_{0,t};$$

(77)

which is obtained from Eq. (12) in the literature [23], if we replace $\underline{u}$ by Δu in (12), multiply the equality obtained by h^2, and discard the remainder term.

Then we can write

$$48 U_{0,t} = 4 \sum_{i=1}^{6} U_{i,t} + 2 \sum_{i=7}^{18} U_{i,t} - 6h^2 \Delta U_{0,t} - h^2 \sum_{i=1}^{6} \Delta U_{i,t}.$$

We substitute $t + l$ in place of $\underline{t}$ in this formula, and the formula obtained to the original formula, replace $\Delta u_{k,t}(k = 0, 1,\ldots,6)$ by $a^{-2}u'_t(x_k, y_k, z_k, t)$ in the sum, and use an equality simi$\cdot$ $^-$ to (32) written for three spatial coordinates.

We thus obtain the relationship

$$(48 + 12\lambda) U_{0,t+l} = (4 - 2\lambda) \sum_{i=1}^{6} U_{i,t+l} + 2 \sum_{i=7}^{18} U_{i,t+l} +$$

$$+ (4 + 2\lambda) \sum_{i=1}^{6} U_{i,t} + 2 \sum_{i=7}^{18} U_{i,t} - (48 - 12\lambda) U_{0,t}$$

and another

$$(56 + 12\lambda) U_{0,t+l} = (8 - 2\lambda) \sum_{i=1}^{6} U_{i,t+l} + \sum_{r=19}^{26} U_{r,t+l} +$$

$$+ (8 + 2\lambda) \sum_{i=1}^{6} U_{i,t} + \sum_{r=19}^{26} U_{r,t} - (56 - 12\lambda) U_{0,t},$$

constructed with the aid of (74) and (77).

The number of these formulas may easily be increased by using the equalities presented in the literature [23].

For example, we add to these formulas one more formula

$$(224 - 80\lambda) U_{0, i+i} = 32 \sum_{i=1}^{6} U_{i, i+i} + (4 - 2\lambda) \sum_{r=19}^{26} U_{r, i+i} + \qquad (78)$$

$$+ (4 + 2\lambda) \sum_{r=19}^{26} U_{r, i} + 32 \sum_{i=1}^{6} U_{i, i} - (224 - 80\lambda) U_{0, i}$$

which, when $\lambda = \dfrac{h^2}{la^2} = 2$, assumes the form

$$48 U_{0, i+i} = 4 \sum_{i=1}^{6} U_{i, i+i} + \sum_{r=19}^{26} U_{r, i} + 4 \sum_{i=1}^{6} U_{i, i} - 8 U_{0, i}. \qquad (79)$$

Formula (78) is derived in the same way as the preceding two by using (74) and the equality

$$h^2 \sum_{r=19}^{26} \Delta U_{r, i} = 8 h^2 \Delta U_{0, i} + 4 h^4 \Delta^2 U_{0, i}$$

which we find using Eq. (17) from the literature [23].

Formula (78) leads to a system of equations which may be solved by the method of successive approximations. Convergence takes place for values of λ satisfying the inequality

$$\frac{4}{3} < \lambda \leq 2.$$

We arrive at this inequality by following, in a general way, the discussions presented above for the two-dimensional case.

For this purpose, let us consider a cube Q_1 with faces parallel to the coordinate planes and located at a distance h from the boundary under the assumption that Q_1 lies wholly inside the given cube for which Eq. (72) is solved. Network nodes lying on Q_1 shall be called first-proximity nodes. Next, we take a cube (lying wholly inside Q_1) again with faces parallel to the coordinate planes and located at a distance h from the corresponding faces of Q_1, and the nodes

lying on it shall be called second-proximity nodes, etc. It remains to make estimates of the error in the solution along the cubes Q_1, Q_2, ..., on which the nodes lie, in exactly the same way as we estimated the error along the squares γ_1, γ_2,... above.

We shall now solve an example which explains how the method that has been set forth is applied to physical problems. Assume that a homogeneous cube of cast iron with side L = 1 m is being cooled and that a temperature of 0°C is maintained on all the faces of the cube throughout the entire cooling process. It is required to find the temperature distribution inside the cube at the moment of time T = 1.6 hrs, when the initial temperature is distributed (inside the cube) in the following way:

$$u(x, y, z, 0) = 50 \sin \pi x \sin \pi y \sin \pi z. \tag{80}$$

In order to solve our problem, we must integrate Differential Equation (72) with Initial Condition (80) and the boundary condition

$$u = 0 \tag{81}$$

for $t \geqslant 0$.

In determining the thermal diffusivity of cast iron we assume that the thermal conductivity is

$$\lambda = 54 \frac{kcal}{hr \cdot m \cdot °C},$$

the specific heat

$$c = 0.12 \frac{kcal}{kg \cdot °C}, \text{ and}$$

the specific gravity

$$\gamma = 7200 \frac{kg}{m^3},$$

where kcal, as usual, means kilocalorie (large calorie).

Next, let us proceed to calculate the thermal diffusivity. According to the data of the preceding paragraph, we find:

$$a^2 = \frac{\lambda}{c\gamma} = 0,0625 \ \frac{m^2}{hr}.$$

We are now able to integrate Eq. (72) numerically. For this purpose, let us take the length of an edge of a network cube h = 0.2 and use a seven-term equality of type (79), for which the spacing with respect to $\underline{t}$ should satisfy the condition

$$l = \frac{h^2}{2a^2} = 0,32 \ (hours).$$

The initial values of the temperature in the network nodes are calculated from (80).

$\zeta = \frac{1}{5}$

1	2	2	1
2	3	3	2
2	3	3	2
1	2	2	1

$\zeta = \frac{2}{5}$

2	3	3	2
3	4	4	3
3	4	4	3
2	3	3	2

$\zeta = \frac{3}{5}$

2	3	3	2
3	4	4	3
3	4	4	3
2	3	3	2

$\zeta = \frac{4}{5}$

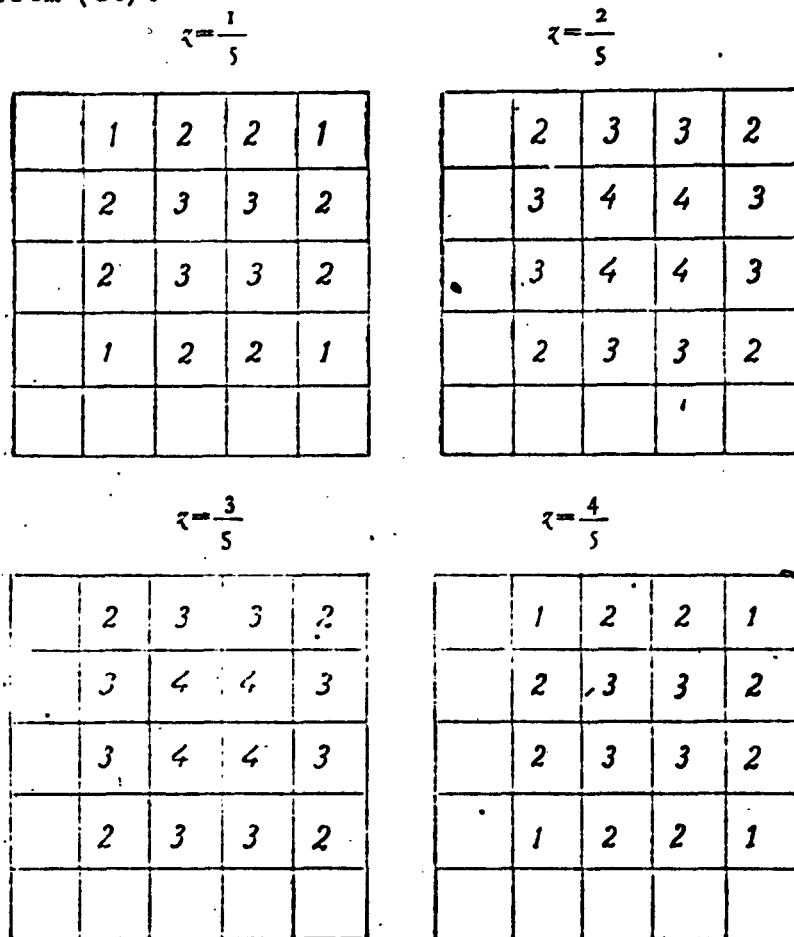| 1	2	2	1
2	3	3	2
2	3	3	2
1	2	2	1

Fig. 2.

In setting up the equations it is advantageous to use the cross sections of the cube cut by the planes

$$z_k = \frac{k}{5}(k = 1, 2, 3, 4).$$

perpendicular to its edges. These cross sections are presented in Fig. 2; in them certain nodes [viz. those in which the values of $u(x, y, z, t)$ coincide due to the symmetrical temperature distribution] are designated by the same numbers.

In the network nodes lying on the boundaries of these squares, as well as on the faces of the cube $z = 0$ and $z = 1$, the temperature at any moment of time should be taken equal to zero, in accordance with (81).

Applying Formula (79) to our example (to the nodes designated in Fig. 2), we obtain a system of linear equations:

$$48\, U_{1,t+l} = 12 U_{2,t+l} - 8 U_{1,t} + 12\, U_{2,t} + U_{4,t},$$
$$44\, U_{2,t+l} = 4\,(U_{1,t+l} + 2\, U_{3,t+l}) + 4 U_{1,t} - 4 U_{2,t} + 9\, U_{3,t} + U_{4,t},$$
$$40\, U_{3,t+l} = 4\,(U_{4,t+l} + 2\, U_{2,t+l}) + 9\, U_{2,t} + 2 U_{3,t} + 5 U_{4,t},$$
$$36\, U_{4,t+l} = 12\, U_{3,t+l} + U_{1,t} + 5 U_{2,t} + 15\, U_{3,t} + 5 U_{4,t},$$

solving which for $t = 0$, l, $2l$, $3l$, and $4l$, we find a solution to Eq. (72) (with coefficient $\alpha^2 = 0.0625$) such that when $t = 0$ it reverts in the network nodes to the given initial state, according to (80).

To avoid the method of successive approximations, it is expedient to first solve the obtained system of equations for the unknowns $U_{k,\ t+l}(k = 1, 2, 3, 4)$, and then perform the computations for each layer using the temperature values in the nodes of the preceding layer. We obtain the following four formulas.

$$14388\, U_{1,t+l} = -\,2121\, U_{1,t} + 3504\, U_{2,t} + 849\, U_{3,t} + 494\cdot U_{4,t}\,,$$

$$14388\, U_{2,t+l} = 1168\, U_{1,t} - 372\, U_{2,t} + 3396\, U_{3,t} + 777\, U_{4,t}\,,$$

$$14388\, U_{3,t+l} = 283\, \Gamma_{1,t} + 3396\, U_{2,t} + 2067\, U_{3,t} + 2228\, U_{4,t}\,,$$

$$14388\, U_{4,t+l} = 494\, \Gamma_{1,t} + 2331\, U_{2,t} + 6684\, U_{3,t} + 2741\, U_{4,t}\,.$$

The results of calculations using these formulas are given in Table 3.

TABLE 3

Temperature values in nodes of cubic network calculated according to formula (79)

t	Approximate values				Exact values			
	$U_{1,t}$	$U_{2,t}$	$U_{3,t}$	$U_{4,t}$	$U_{1,t}$	$U_{2,t}$	$U_{3,t}$	$U_{4,t}$
0	10,15	16,43	26,58	43,01	10,15	16,43	26,58	43,01
l	5,55	9,00	14,56	23,55	5,62	9,09	14,70	23,79
2l	3,04	4,93	7,97	12,90	3,11	5,03	8,13	13,16
3l	1,67	2,70	4,37	7,06	1,72	2,78	4,50	7,28
4l	0,91	1,48	2,39	3,87	0,95	1,54	2,49	4,03
5l	0,50	0,81	1,31	2,12	0,53	0,85	1,38	2,23

The exact values of $u_{1,kl}$ in the node $(x_1,\ y_1,\ z_1)$ for various moments of time $t = kl$ are calculated using the formula

$$u_{1,kl} = 50 \sin \pi x_i \sin \pi y_i \sin \pi z_i\, e^{-0,1875\, kl\, \pi^2} \quad (k=0,\ 1,\ \dots,\ 5),$$

since the function

$$u\,(x,\ y,\ z,\ t) = 50 \sin \pi x \sin \pi y \sin \pi z\, e^{-0,1875\, t\, \pi^2}$$

satisfies the given differential equation, as well as the given initial and boundary conditions.

The method is also applicable in the case of an arbitrary closed surface. It is only necessary to set up special equations for the boundary nodes, as we did for the two-dimensional case. A large number of such equations may be constructed. For example, the simplest of them can be obtained with the aid of the formula:

$$\Delta u_{\nu,t} = 2\left[\frac{1}{h_1 + h_2}\left(\frac{u_{C,t}}{h_1} + \frac{u_{A,t}}{h_2}\right) + \frac{1}{h_3 + h_4}\left(\frac{u_{B,t}}{h_3} + \frac{u_{D,t}}{h_4}\right) + \right.$$
$$\left. + \frac{1}{h_5 + h_6}\left(\frac{u_{F,t}}{h_5} + \frac{u_{E,t}}{h_6}\right) - \left(\frac{1}{h_1 h_2} + \frac{1}{h_3 h_4} + \frac{1}{h_5 h_6}\right)u_{\nu,t}\right],$$

where $u_{\nu,t}$, $u_{C,t}$,..., $u_{E,t}$ are the values of the function $u(x, y, z, t)$ in the main node ν and the nodes next to it C, A,..., E, which are separated from the main node (in directions opposite to or coincident with the directions of the coordinate axes) by a distance of h_1, h_2, ..., h_6, respectively; and $\Delta u_{\nu,t}$ is the value of the Laplace operation for the function $u_{\nu,t}$ in the node with the number ν at the moment $\underline{t}$.

In the case of boundary conditions of the general type it is necessary to have a formula of increased accuracy for the approximation of the normal derivative. For a cube they can be obtained by reasoning in the same way as in item B in the derivation of the formula for a square.

Received December 29, 1959.

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